

# High-resolution distributed sampling of bandlimited fields with low-precision sensors\*

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**Abstract**—The problem of sampling a discrete-time sequence of spatially bandlimited fields with a bounded dynamic range, in a distributed, communication-constrained, processing environment is addressed. A central unit, having access to the data gathered by a dense network of low-precision sensors, operating under stringent inter-node communication constraints, is required to reconstruct the field snapshots to maximum accuracy. Both deterministic and stochastic field models are considered. For stochastic fields, results are established in the almost-sure sense. The feasibility of having a flexible tradeoff between the oversampling rate (sensor density) and the analog-to-digital converter (ADC) precision, while achieving an exponential accuracy in the number of bits per Nyquist-interval per snapshot is demonstrated. This exposes an underlying “*conservation of bits*” principle: the bit-budget per Nyquist-interval per snapshot (the rate) can be distributed along the amplitude axis (sensor-precision) and space (sensor density) in an almost arbitrary discrete-valued manner, while retaining the same (exponential) distortion-rate characteristics. An achievable information scaling law for field reconstruction over a bounded region is also derived. With  $N$  one-bit sensors per Nyquist-interval,  $\Theta(\log N)$  Nyquist-intervals, and total network bitrate  $R_{net} = \Theta((\log N)^2)$  (per-sensor bitrate  $\Theta((\log N)/N)$ ), the maximum pointwise distortion goes to zero as  $D = O((\log N)^2/N)$  or  $D = O(R_{net}2^{-\beta\sqrt{R_{net}}})$ . This is shown to be possible with only nearest-neighbor communication, distributed coding, and appropriate interpolation algorithms. For a fixed, nonzero target distortion, the number of sensors and the network rate needed is always finite.

**Keywords:** bandlimited fields; nonuniform sampling; dithered scalar quantization; oversampled analog-to-digital conversion; distributed source coding; sensor networks; scaling law;

## I. INTRODUCTION

*Motivation:* High resolution remote sensing of physical phenomena is a task of considerable importance in applications such as environment/weather monitoring, ecology, and precision agriculture. Consider the scenario where a large distributed network of low-precision, low-power sensors is deployed over a region of interest to collect and return measurements to a central data collection and processing unit (CPU)<sup>1</sup> over some time duration. The goal is to efficiently

acquire, process, and transport data to faithfully reconstruct the physical field under appropriate measures of fidelity.

Physical fields such as temperature, pressure, magnetization, and vibration are fundamentally analog in nature. Their spatio-temporal distributions obey fundamental physical laws (a temperature field must satisfy Laplace’s equation, the velocity field in fluid flows must satisfy Navier-Stokes equation, etc.) that induce strong dependencies across space and time. Many physical fields are approximately bandlimited in space and time (the physical propagation laws often providing a natural filtering effect that attenuates high frequencies) and can be reconstructed in a stable manner from samples taken slightly above the Nyquist-rate on a uniform lattice. However, in practice, the samples of the field are quantized due to the finite precision of ADCs, leading to unavoidable field reconstruction errors. When bandlimited fields are uniformly sampled at the critical Nyquist-rate (in space and time), the worst-case pointwise reconstruction error  $D(R)$  decays exponentially with the bitrate  $R$  (measured in number of bits per Nyquist-interval per snapshot<sup>2</sup> of the ADCs [1]–[5], that is,  $D(R)$  is of the order of  $O(2^{-R})$ ). Error can be reduced by increasing the quantization resolution of the sensors. Is it possible to sacrifice ADC-precision (lower amplitude resolution) for denser spatial sampling rate (higher spatial resolution) while maintaining the same reconstruction quality and the same number of information bits per Nyquist-interval per snapshot? In the sequel we show that it is indeed possible to achieve a flexible array of tradeoffs between the (amplitude) resolution and the density (spatial resolution) of sensors leading to a “*conservation of bits*” principle.

The classical problem of sampling of continuous signals is a mature topic in signal processing that has accumulated a rich knowledge-base over the past several decades [1], [2], [6]. However, sensor networks impose challenging constraints on the classical sampling paradigm in terms of low device precision and power and prohibitive communication costs. This biases solutions toward highly distributed and/or localized collaborative processing environments characterized by nearest-neighbor communication. This also calls for distributed compression algorithms to be integrated with sampling. We use the term “distributed sampling” to capture these effects. This paper is accordingly driven by the goal of addressing the sampling problem within the context of sensor networks and their associated constraints in terms of both localized processing and communication. At an information processing

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<sup>1</sup>Sensors can take turns to be the CPU.

<sup>2</sup>Sensors are assumed to periodically sample the field in synchronism at discrete time instants at their locations. A field snapshot refers to the entire continuous-space field at one sampling time instant.

level, there are two broad functional tasks in data-gathering sensor networks: (i) data-acquisition or sampling, processing, and coding of the sensor field measurements and (ii) data transport, where the acquired data is disseminated across the network to a CPU. Both tasks are addressed in this work. The data-acquisition task is addressed through a distributed sampling framework that is well-suited for sampling sensor fields with minimal or no inter-sensor communication. The data-transport task is addressed through distributed coding and nearest-neighbor communication that moves the sensor data to the CPU. For a related information-theoretic study of multiuser source coding with distributed vector quantization, see [7], [8]. For work with a different flavor on quantization and oversampling, see [9].

*Contributions:* (i) “*Bit-conservation*” principle: A key contribution of this work is that the bit-budget per Nyquist-interval (the rate) can be distributed along the amplitude axis (sensor-precision) and space (sensor density) in an almost arbitrary discrete-valued manner, while retaining (the best-known) exponential distortion-rate characteristics (Section VI). This provides the network-architect with a guiding design principle for *spatially adaptive sampling* in terms of selecting sensor precision, density, and deployment patterns to meet the desired reconstruction quality. We can use higher precision ADCs when the sampling density is constrained to be light by the underlying topography of sensor-deployment, e.g., rugged terrain or occluding obstacles, and use lower precision sensors when the sampling density can be high. This also clarifies the need for *diversity* in sensing capabilities for easing the adaptation to changing environments.

(ii) *Information scaling law*: The pioneering work of Gupta and Kumar [10] that has precipitated interest in scaling laws for ad hoc networks, applies to scenarios where nodes produce *independent data*, no matter what the scale of the network. This does not entirely apply to the sensor network context, where the “information density” (related to the sensed field) remains fixed regardless of the “network density” (related to the network size). The inter-sensor data correlation induced by the underlying physics of the phenomenon being sensed can be exploited to reduce the net information to be disseminated in the network. How does network information grow with increasing demands on quality? What are the network resources needed to sustain high-quality large-scale field acquisition? How should the underlying spatio-temporal correlation be exploited? For reconstructing a discrete-time sequence of spatially bandlimited fields over a fixed bounded region at a CPU, we show that the following scaling behavior can be realized with one-bit sensors, distributed coding, neighbor-to-neighbor local communication, and suitable spatial interpolation (Section VII-B.2): With  $N$  one-bit sensors per Nyquist-interval,  $\Theta(\log N)$  Nyquist-intervals, and total network bitrate  $R_{\text{net}} = \Theta((\log N)^2)$  (per-sensor bitrate  $\Theta((\log N)/N)$ ), the maximum pointwise distortion goes to zero as  $D = O((\log N)^2/N)$  or  $D = O(R_{\text{net}} 2^{-\beta \sqrt{R_{\text{net}}}})$ .

(iii) *Stochastic fields*: An important technical contribution of this work is the generalization of the nonuniform sampling results for amplitude limited, bandlimited, square-integrable deterministic fields due to Cvetković and Daubechies [11] to

amplitude limited, bandlimited, wide-sense stationary (WSS) spatial stochastic processes (Sections III, IV-B, V-B, and VI-B). This generalization is effected by leveraging certain results for the Zakai class of bandlimited fields due to Zakai [12] and Cambanis and Masry [13]. The maximum pointwise error versus rate results are established in the strong almost-sure sense (hence also in the expected  $p$ -th power sense,  $p \in (0, \infty)$ ). We build upon the work by Cvetković and Daubechies along three directions. First, we generalize their results to arbitrary precision ADCs. Secondly, we extend their framework to the stochastic setting. Thirdly, we leverage these results from the classical centralized setup to the distributed setting accounting for information transport costs making them relevant to the sensor network context.

*Key ideas*: A key concept underlying the ability to achieve a flexible tradeoff between amplitude and spatial resolution is dithered sampling due to Cvetković and Daubechies [11]. For each field snapshot, sensors add (or compare) the value of a pre-designed dither function at their respective locations to their observations and note only the *sign* of the sum. By design of the dither function, the *sum* of the bandlimited field and the dither function is ensured to have *exactly one zero-crossing in every Nyquist-interval*. This induces a strong dependency in the binary observations of all  $N = 2^R$  sensors inside each Nyquist-interval. Their joint entropy-rate is ensured to be not more than  $R$  bits per snapshot. This data can be compressed using a distributed source code and moved to the CPU using nearest-neighbor communication. The total rate for this is not more than  $R$  bits per Nyquist-interval per snapshot. The *spatial resolution* of the reconstructed zero-crossings, which is proportional to inter-sensor separation  $= O(2^{-R})$ , can be translated to *amplitude resolution* of the reconstructed field at these locations through the local smoothness properties of the field and the dither (mean-value theorem). Using results in nonharmonic Fourier analysis due to Cvetković and Daubechies [11], per-sample amplitude resolution can be converted, as will be shown, to global field approximation accuracy of the same order using interpolation techniques. The maximum reconstruction error using  $2^R$  one-bit sensors per Nyquist-interval will then be of the order of  $O(2^{-R})$ . This is equivalent to the reconstruction accuracy of one  $R$ -bit sensor per Nyquist-interval. A similar approach using dithered *level*-crossings instead of zero-crossings allows one to increase precision and decrease density maintaining order-optimal accuracy.

*Caveats*: This work has a sampling-theoretic focus. Although the motivation is two-dimensional sensor networks, for clarity, the theoretical development focuses on a single time-snapshot and spatial dimension equal to one. The multidimensional case is discussed in the last section. We also acknowledge that there are several important problems that must be overcome to make the results of this work practically viable. This includes sensing-noise, sensor-deployment issues, sensor location errors or uncertainties, and sensor synchronization and scheduling issues. We defer the exploration of such issues to future research.

*Organization*: Field modeling assumptions, distortion criteria, and assumptions on data-acquisition and communication,

are discussed in Section II. Section III discusses the key technical issues and results in going from the deterministic to the stochastic setup. Throughout, the deterministic and the stochastic scenarios are discussed side-by-side to emphasize similarities and differences. Classical deterministic Nyquist sampling and its stochastic extension is discussed in Section IV. Section V discusses single-bit dithered oversampling, first by summarizing the deterministic results by Cvetković and Daubechies [11] and then extending it to the stochastic case. A “*bit-conservation*” principle and its implications are exposed in Section VI by showing how a flexible trade-off between sensor-precision and sensor density can be achieved. Distributed processing and communication issues and costs germane to sensor networks are discussed in Section VII. This includes distributed field-acquisition and coding (Section VII-A), information transport and associated achievable information scaling law in terms of the bitrate, distortion, and sensor density for field reconstruction in a compact region of interest (Section VII-B.2), and extensions to two and higher spatial dimensions (Section VII-C). The main contributions are summarized in Section VIII. Proofs of technical results are relegated to the appendices to maintain a smooth flow of ideas.

*Notation:* Real numbers, the integers, and the natural numbers are respectively denoted by  $\mathbb{R}$ ,  $\mathbb{Z}$ , and  $\mathbb{N}$ . The term ‘field’ is used to emphasize the dependence of a quantity on *spatial* coordinates, e.g., electromagnetic field, temperature field, pressure field, etc. All random variables and processes are defined with respect to a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $\mathbb{E}[\cdot]$  denotes the mathematical expectation operator. Random quantities are denoted by capital letters (e.g.,  $X$ ) and specific realizations of random quantities by small letters (e.g.,  $x$ ). Modifications of  $f(t)$  and  $X(t)$ , e.g.,  $f'$ ,  $X'$ ,  $\hat{f}$ ,  $\hat{X}$ , etc., respectively denote deterministic and stochastic fields and  $t, T, t_l, T_l$ , etc., are used for spatial (not temporal) variables. The notation  $X$  without adornments denotes the *entire* process  $\{X(t)\}_{t \in \mathbb{R}}$ .  $\mathcal{L}^p(\mathbb{R}^d)$  denotes the space of  $p$ -th power ( $p \in (0, \infty)$ ), Lebesgue-integrable fields on  $\mathbb{R}^d$  where  $d = \text{dimension}$ .  $BL(S)$  denotes the space of deterministic, square-integrable (finite-energy), continuous, bandlimited fields<sup>3</sup> on  $\mathbb{R}^d$ , with a compact spectral support set  $S$ . The phrase ‘almost-surely’ is synonymous with the phrase ‘with probability one’. The term bitrate is used to denote bits per spatial Nyquist-interval/area/volume for each field snapshot. We follow Landau’s asymptotic notation:  $f(x) = O(g(x))$  as  $x \rightarrow a \Leftrightarrow \limsup_{x \rightarrow a} |f(x)/g(x)| < \infty$ ;  $f(x) = \Theta(g(x)) \Leftrightarrow f(x) = O(g(x))$  and  $g(x) = O(f(x))$ . The value of  $a$  is either  $\infty$  or 0 and will always be clear from the context.

## II. MODELING ASSUMPTIONS AND FIDELITY CRITERIA

We consider a discrete-time sequence of continuous-space spatially bandlimited fields with a limited amplitude range. Both deterministic and stochastic field models, described below in Section II-A, are studied. These modeling assumptions

hold for each discrete-time field snapshot with spatial dimension  $d$  equal to one. An extension to higher spatial dimensions is discussed in Section VII. When dependencies across time snapshots are arbitrary, our error versus rate results apply only to each snapshot and not to other intermediate time instants. If the field is temporally bandlimited, snapshots are assumed to be taken at the temporal Nyquist-rate. Then our results apply to all spatio-temporal points. Knowledge of the field model and associated parameters such as bandwidth and dynamic range are assumed to be available during the design-phase prior to sensor deployment. Sensors are assumed to be deployed in a uniform rectangular grid and the sensor locations are assumed to be available to the fusion center. Sensors are assumed to operate in a time-synchronous manner. Although joint source-channel issues and power-distortion considerations are not considered in this work, the data-transport costs in terms of bitrate are studied in Section VII. The finite-rate encoded sensor observations together with their sensor identification labels are made available to a fusion center. The objective of the fusion center is to estimate the field value at each continuous-space location for each discrete-time snapshot. The estimation quality is measured by the maximum absolute error over all space and time. Bounds on these maximum errors over all space and time also imply bounds on related space and time *averaged* distortion criteria described in Section II-B.

### A. Field-models

The following modeling assumptions apply to *each* temporal snapshot of the field that is acquired by the sensor network. For clarity, the theoretical development focuses on a single time-snapshot and spatial dimension equal to one. The multi-dimensional and temporal aspects are discussed in Section VII.

1) *Deterministic case:* We assume that  $f(t)$ ,  $t \in \mathbb{R}$ , is an amplitude limited field, with amplitude limit  $A$ , belonging to  $BL([-W, W])$ . Here, both  $A$  and  $W$  are some finite, strictly positive, real numbers. Distributed sampling of deterministic, square-integrable, *non-bandlimited* fields, with an exponentially decaying spectrum, can also be studied using the techniques discussed here (see [14] for details), but will not be entered into. It should be noted that a limited amplitude constraint is less restrictive than a limited energy constraint for bandlimited fields. A bandlimited field with limited energy is necessarily amplitude limited but the converse need not be true, e.g.,  $\forall t \in \mathbb{R}$ ,  $f(t) = \sin(t)$ . The modeling assumptions admit fields with an arbitrarily high but finite energy.

A property of bandlimited fields that plays an important role in the derivation of field-interpolation error bounds from nonuniform samples [11], and used in our work, is the uniformly bounded slope property. Specifically, according to Bernstein’s inequality [15], [16, p. 144], for fields in  $BL([-W, W])$  with amplitude limit  $A$ ,

$$|f'(t)| \leq W \sup_{t \in \mathbb{R}} |f(t)| \leq WA, \quad \forall t \in \mathbb{R}. \quad (2.1)$$

Through a suitable renormalization of the amplitude and spatial axes, it can be assumed, without loss of generality, that the amplitude ranges of  $f$  and  $f'$  are respectively contained in

<sup>3</sup> In  $\mathcal{L}^2(\mathbb{R}^d)$  a bandlimited field can be discontinuous on a measure-zero set that may even be uncountably infinite. The continuity condition is explicitly included due to our interest in sample-interpolated reconstructions.



$[-1, 1]$  and  $[-\pi, \pi]$  and the spectral support of  $f$  is contained in  $[-\pi, \pi]$ . This corresponds to  $W = \pi$  and  $A = 1$ .

2) *Stochastic case:* We assume that  $X(t), t \in \mathbb{R}$ , is an amplitude limited stochastic process (field), with amplitude limit  $A$ , which is WSS with an autocorrelation function  $R_X(t), t \in \mathbb{R}$ , belonging to  $BL([-W, W])$ . Note that the process  $X$  is mean-square continuous since  $R_X$  is continuous.<sup>4</sup> Although somewhat restrictive, the class of processes which are stationary, bandlimited, and amplitude limited, is quite rich. Some examples are presented in Appendix I.

### B. Distortion criteria

The performance of different sampling and reconstruction schemes will be compared by deriving a rate-dependent upper bound for the maximum (over space) field reconstruction error, that is, the  $\mathcal{L}^\infty$ -norm of the error.

1) *Deterministic case:* Let  $\hat{f}_R(t)$  denote the reconstruction of a single field snapshot  $f(t)$  at the spatial location  $t \in \mathbb{R}$ , according to some sampling, quantization, and reconstruction scheme which uses  $R$  bits per meter per snapshot. Let  $e(t) = (f(t) - \hat{f}_R(t))$  denote the reconstruction error at  $t$ . For different sampling and reconstruction schemes, we will derive upper bounds  $\eta(R)$ , on the maximum (over space  $t$ ) field reconstruction error,  $D(R) := \|e\|_\infty$ . These bounds then automatically hold for the so-called *locally-averaged*<sup>5</sup>  $\mathcal{L}^q$  quasi-norm<sup>6</sup> of the reconstruction error  $e$  defined by

$$\|e\|_q(t, T) := \left\{ \frac{1}{T} \int_{|s-t| < T/2} |e(s)|^q ds \right\}^{\frac{1}{q}}, \quad q \in (0, \infty),$$

because for all  $t \in \mathbb{R}$  and all  $T \in (0, \infty)$ ,  $\|e\|_q(t, T) \leq \|e\|_\infty$ .

2) *Stochastic case:* Let  $\hat{X}_R$  and  $E = (X - \hat{X}_R)$  respectively denote the reconstruction and error fields according to some sampling, quantization, and reconstruction scheme which uses  $R$  bits per meter per snapshot. It should be noted that  $E$  may not be WSS because the reconstruction scheme may be nonlinear (due to quantization) and spatially varying (due to sampling). For different sampling and reconstruction schemes, we will derive upper bounds  $\eta(R)$  such that  $\|E\|_\infty \leq \eta(R)$  almost surely and consequently also  $\|(\mathbb{E}|E|^p)^{1/p}\|_\infty \leq \eta(R)$  for any  $p \in (0, \infty)$ . As in the deterministic case, these bounds will also apply to the locally-averaged error ( $\|E\|_q(t, T)$ ) both in the almost-sure and expected  $p$ -th power senses.

## III. FROM DETERMINISTIC TO STOCHASTIC FIELDS

To expose the intuition underlying the main results of this work and for expositional clarity, in the sequel, we first discuss results for the deterministic setting, and then immediately extend this to the stochastic setting. Moving from the deterministic to the stochastic setting would have been straightforward if almost all sample paths of  $X$  inherited the same

properties assumed for deterministic fields. The deterministic nonuniform sampling and interpolation results for amplitude and bandlimited fields by Cvetković and Daubechies [11], that we build upon, are stated for square-integrable fields that also satisfy a crucial uniformly bounded slope property (Bernstein's inequality (2.1)). Moreover, the nonuniform sampling locations in these deterministic results, are intricately coupled to field values, because they are derived from certain level-crossings of the field. It is unclear if these conditions hold in the stochastic setting. For instance, while every sample path of  $X$  is amplitude limited, it need not be bandlimited in the conventional sense. In fact, the sample paths need not even be square-integrable,<sup>7</sup> which is a common assumption in results pertaining to the approximation of deterministic fields from a discrete set of samples. The extension of sampling results from the conventional square-integrable bandlimited fields to the stochastic case is one of the key contributions of our work.

It turns out that almost all sample paths of a mean-square continuous, WSS stochastic process, with an autocorrelation function in  $BL([-W, W])$ , are bandlimited in the sense of Zakai [12], [13], [18]. This is an extension of the conventional notion of bandlimited fields. The Zakai class of bandlimited fields (defined below) includes the conventional square-integrable bandlimited fields but also includes fields such as nonzero constants and pure tones (sinusoids) which are not square-integrable. Therefore, our approach for extending sampling results from the deterministic to the stochastic setting is to first establish deterministic sampling results for the Zakai class of bandlimited fields, which includes non square-integrable fields, via Lemma 3.1. The results for the stochastic setting follow immediately, in a strong almost-sure sense, via Fact 3.1 below. The results then automatically hold in an expected  $p$ -th power sense as well.

*Definition 3.1: (Zakai class of bandlimited fields [12], [13], [18] ZBL( $W, \delta$ ))* A field  $f(t)$ ,  $t \in \mathbb{R}$ , is said to be bandlimited in the sense of Zakai, with bandwidth  $W$  and a margin parameter  $\delta \in (0, \infty)$ , if  $\int_{-\infty}^{\infty} |f(t)|^2 \frac{1}{1+t^2} dt < \infty$  and

$$f(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau, \quad (3.1)$$

where  $h(t) = \frac{2}{\pi \delta t^2} \sin((W + \delta/2)t) \sin(\delta t/2)$  for  $t \neq 0$  and  $h(0) := (2\pi)^{-1} (2W + \delta)$ .  $ZBL(W, \delta)$  is the set of all fields which are bandlimited in the sense of Zakai with bandwidth  $W$  and margin parameter  $\delta$ .

To provide some insight into the Zakai class of bandlimited fields, we summarize some of its salient properties in this paragraph. These properties are not needed in the subsequent development. The function  $h$  appearing in the definition of  $ZBL(W, \delta)$  has a bounded amplitude and is both absolutely and square integrable because it decays sufficiently fast (as  $1/t^2$ ). It has a Fourier transform given by the trapezoidal-shaped function shown in Figure 1. Thus  $h \in BL([-W - \delta, W + \delta])$ . The set of all Borel-measurable complex-valued functions  $f(t)$  on the real line satisfying  $\int_{-\infty}^{\infty} |f(t)|^2 \frac{1}{1+t^2} dt < \infty$

<sup>7</sup>If  $X(t)$  is a strictly stationary ergodic process with  $0 < R_X(0)$ , then almost-surely  $R(0) = \lim_{T \rightarrow \infty} (2T)^{-1} \int_{[-T, T]} X^2(t) dt$ . Thus almost all sample paths are not square-integrable.

<sup>4</sup>Continuity at  $t = 0$  suffices since it implies continuity  $\forall t \neq 0$  [17].

<sup>5</sup>This is a deterministic spatial average and not a stochastic average with respect to any probability distribution.

<sup>6</sup>Note that this is a norm only for  $q \in [1, \infty)$ . We do not require this to be a norm so we allow  $q \in (0, \infty)$ .

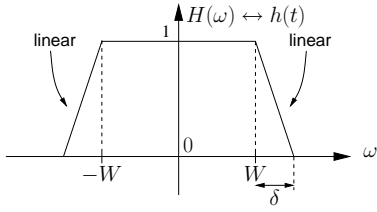


Fig. 1. **Fourier transform of kernel  $h(t)$ .**

$\infty$  forms a Hilbert space with inner product  $\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g^*(t) \frac{1}{1+t^2} dt$  where  $*$  denotes complex conjugate. This Hilbert space includes  $\mathcal{L}^2(\mathbb{R})$ , bounded functions, and functions for which  $\frac{1}{2T} \int_{-T}^T |f(t)|^2 dt$  is bounded in  $T$ , e.g.,  $f(t) = \sin(t)$ ,  $t \in \mathbb{R}$  [12].  $ZBL(W, \delta)$  then forms a reproducing-kernel subspace of this Hilbert space with the Toeplitz reproducing kernel  $h$ . Every function in  $ZBL(W, \delta)$  is equal almost everywhere (with respect to the Lebesgue measure) to a continuous function (given by the right side of (3.1)) and, as in the deterministic case (see footnote 3), only these continuous representatives will be considered. For all  $\delta, W$ ,  $BL([-W, W]) \subseteq ZBL(W, \delta)$ , that is, conventionally (square-integrable) bandlimited fields are also Zakai-sense bandlimited. It is possible to use other kernels for  $h$ , e.g., kernels whose Fourier transforms increase and decrease super-linearly over  $[-W - \delta, -W]$  and  $[W, W + \delta]$  respectively. Extending prior work due to Zakai [12], in [13] Cambanis and Masry showed that for all  $W, \delta$ , a field  $g \in ZBL(W, \delta)$  if and only if, for all  $t \in \mathbb{R}$ ,  $g(t) = g(0) + tf(t)$ , for some  $f \in BL([-W, W])$ . Thus,  $ZBL(W, \delta)$  is in fact independent of the value of the margin parameter  $\delta$  which can be taken to be any arbitrarily small but strictly positive real number and mainly serves to simplify convergence issues.

The following uniformly bounded slope property for amplitude limited fields in  $ZBL(W, \delta)$ , proved in Appendix II, is crucial for establishing the asymptotic exponential decay of approximation error with bitrate for the stochastic case. This is discussed in Sections V and VI in the context of sampling using low-precision ADCs. The bound provided is sufficient for our purpose although it could potentially be strengthened to something more like Bernstein's inequality (2.1).

**Proposition 3.1:** (Amplitude limited fields in  $ZBL(W, \delta)$  have bounded slope) Let  $0 < W, \delta, A < \infty$ ,  $f \in ZBL(W, \delta)$  continuous,  $|f(t)| \leq A$  for all  $t \in \mathbb{R}$ , and  $h$  be as in Definition 3.1. Then  $f$  is differentiable and  $|f'(t)| \leq 2AW^2 < \infty$  for all  $t \in \mathbb{R}$ .

The following key result, proved in Appendix III, shows that general uniform/nonuniform sampling and interpolation results which hold for amplitude limited fields in  $BL([-W, W])$ , will also continue to hold for amplitude limited fields in  $ZBL(W, \delta)$ , for all  $W, \delta$ .

**Lemma 3.1:** (Lifting sampling and interpolation results from  $BL([-W, W])$  to  $ZBL(W, \delta)$ ) Let  $0 < W, \delta, A < \infty$  and  $h \in BL([-W - \delta, W + \delta])$  be the Toeplitz reproducing kernel in Definition 3.1. Let  $\mathcal{T} = \{t_l\}_{l \in \mathbb{Z}}$ , a set of sampling locations, and  $\{\psi_l\}_{l \in \mathbb{Z}}$  a set of interpolation kernels be such that (i)  $C := \sup_{t \in \mathbb{R}} [\sum_{l \in \mathbb{Z}} |\psi_l(t - t_l)|] < \infty$  and (ii) for all

$(t, \tau) \in \mathbb{R}^2$ ,

$$h(t - \tau) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L h(t_l - \tau) \psi_l(t - t_l)$$

(the existence of such  $\mathcal{T}$  and  $\{\psi_l\}_{l \in \mathbb{Z}}$  is ensured by Fact 5.1). Then for any  $f \in ZBL(W, \delta)$  with amplitude limit  $A$ ,

$$f(t) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L f(t_l) \psi_l(t - t_l),$$

where the above series converges absolutely for each  $t$  and uniformly on all compact subsets of  $\mathbb{R}$ .

Sampling and interpolation results which hold for fields in  $ZBL(W, \delta)$ , also hold true for the sample paths of a WSS process with an autocorrelation function in  $BL([-W, W])$ , in the almost-sure sense, due to the following fact.

**Fact 3.1:** (Sample paths of mean-square continuous bandlimited WSS processes [12, Sec. 5], [13, Sec. 3]) Let  $0 < W, \delta < \infty$ . If  $X$  is a WSS stochastic process with an autocorrelation function  $R_X \in BL([-W, W])$ , then almost-surely (with probability one), its sample paths belong to  $ZBL(W, \delta)$ .

Thus, if there is an  $R$  bits per meter per snapshot sampling, quantization, and reconstruction scheme such that  $\|f - \hat{f}_R\|_{\infty} \leq \eta(R)$  for all  $f \in ZBL(W, \delta)$  then for any WSS  $X$  with  $R_X \in BL([-W, W])$ , almost-surely,  $\|X(t) - \hat{X}_R\|_{\infty} \leq \eta(R)$ . Thus an upper bound on the maximum error that holds for deterministic fields which are Zakai-sense bandlimited, also holds for bandlimited WSS stochastic fields in a strong almost-sure sense.

As in the deterministic field model, through a suitable renormalization of the amplitude and spatial axes, it can be assumed, without loss of generality, that the amplitude ranges of  $f$  and  $f'$  are respectively contained in  $[-1, 1]$  and  $[-\pi, \pi]$  and the spectral support of  $f$  is “essentially” contained in  $[-\pi, \pi]$ . This corresponds to  $W = \pi$  with  $\delta$  arbitrarily small but fixed, and  $A = 1/(2W^2)$ .

#### IV. NYQUIST-SAMPLING (IMPROVING QUALITY THROUGH SENSOR-PRECISION)

##### A. Deterministic case

Any field  $f \in BL([- \pi, \pi])$  can be reconstructed from the samples  $\{f(l)\}_{l \in \mathbb{Z}}$  according to the well known interpolation formula [1]

$$f(t) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L f(l) \text{sinc}(t - l), \quad \forall t \in \mathbb{R}, \quad (4.1)$$

where  $\text{sinc}(t) := (1/\pi t) \sin(\pi t)$  for  $t \neq 0$  and  $\text{sinc}(0) := 1$ . The Nyquist-sampling period for  $f$  is  $T_{NQ} = 1$ . However, in practice the reconstruction (4.1) is not stable to bounded perturbations in the sample values due to the poor decay properties of the sinc interpolation kernel, that is, the series in (4.1) is not absolutely-convergent. All practical ADCs have finite precision and the sampling process is invariably subject to perturbations. The instability in the reconstruction implies that the quantization noise/error can potentially build up and

lead to unbounded reconstruction errors in parts of the field.<sup>8</sup> However, the instability can be overcome by taking samples slightly above the Nyquist-rate:

**Fact 4.1:** (*Stable interpolation of bandlimited fields from samples at uniformly spaced locations [11]*) For each  $\lambda > 1/T_{NQ} = 1$ , there exists an absolutely-integrable kernel  $\phi_\lambda(t)$  belonging to  $BL([- \pi\lambda, \pi\lambda])$  such that  $C := \sup_{t \in \mathbb{R}} (\sum_{l \in \mathbb{Z}} |\phi_\lambda(t - (l/\lambda))|) < \infty$  and for all fields  $f \in BL([- \pi\lambda + \delta, \pi\lambda - \delta])$  with  $0 < \delta < \pi\lambda$ ,

$$f(t) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L f(l/\lambda) \phi_\lambda \left( t - \frac{l}{\lambda} \right), \quad \forall t \in \mathbb{R}, \quad (4.2)$$

where the above series converges absolutely for each  $t$  (and in  $\mathcal{L}^2$ ) and uniformly on all compact subsets of  $\mathbb{R}$ .

It is due to the finiteness of  $C$  that the reconstruction series (4.2) is absolutely and uniformly convergent. An example of an interpolation kernel  $\phi_\lambda$  mentioned in Fact 4.1 is the kernel  $h$  in Definition 3.1 with  $W = \pi$  and  $\delta = \pi(\lambda - 1)$ . In fact, there exist interpolation kernels  $\phi_\lambda$  that decay faster than  $c_l t^{-l}$  for all  $l \in \mathbb{N}$  and some constants  $\{c_l\}_{l \in \mathbb{N}}$ . Technically,  $\mathcal{T} = \{t_l := l/\lambda\}_{l \in \mathbb{Z}}$  is said to form a set of stable sampling points for fields in  $BL([- \pi, \pi])$  [2]. This is because bounded perturbations in the sample values at these locations never lead to unbounded reconstruction errors at any point unlike in (4.1) where the sampling interval is exactly equal to a Nyquist-interval.

A “near-Nyquist” stable sampling rate  $\lambda > 1$  but close to one will be held fixed for the rest of this paper and the term oversampling will be used to refer to uniform sampling at a spatial sampling rate which is strictly greater than  $\lambda$ . The term “Nyquist-interval” shall also be used, loosely, to refer to any stable sampling interval of the form  $[l/\lambda, (l+1)/\lambda]$ ,  $l \in \mathbb{Z}$ . The structure of Nyquist-sampling (for sampling rate  $\lambda$ ) is depicted in Figure 2. To study the effect of finite precision ADCs on the

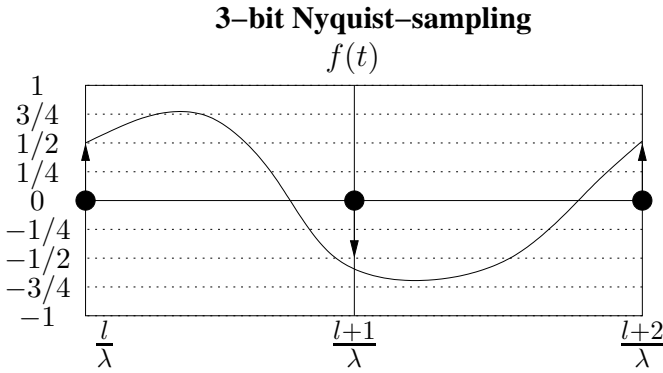


Fig. 2. **Classical Nyquist-sampling with 3-bit ADCs placed at Nyquist-locations  $\{l/\lambda\}_{l \in \mathbb{Z}}$ .** The entire budget of  $k = 3$  bits is exhausted at a single sample point in any Nyquist-interval. The spatially maximum interpolated reconstruction error decays exponentially in the bits per Nyquist-interval  $R = k\lambda$  with exponent  $(1/\lambda)$ .

reconstruction of unit amplitude limited fields in  $BL([- \pi, \pi])$ , let  $Q_k$  denote the  $k$ -bit uniform scalar quantization operation

<sup>8</sup>However, if the perturbations are only due to scalar quantization where  $0$  is a reproduction point, then the interpolation error will be bounded for fields in  $BL([- \pi, \pi])$ .

on  $[-1, 1]$  [19] so that  $|z - Q_k(z)| \leq 2^{-k}$  for all  $z \in [-1, 1]$ . The bitrate in bits per spatial Nyquist-interval for each field snapshot used to quantize the field is given by  $R = k\lambda$ . Let the bitrate- $R$  quantized Nyquist-reconstruction of unit amplitude limited  $f \in BL([- \pi, \pi])$  be defined as:

$$\hat{f}_R^{\text{NQ}}(t) := \lim_{L \rightarrow \infty} \sum_{l=-L}^L Q_k(f(l/\lambda)) \phi \left( t - \frac{l}{\lambda} \right).$$

Using (4.2) in Fact 4.1 and the triangle-inequality, it immediately follows that the spatially maximum reconstruction error can be bounded as

$$D(R) := \|f - \hat{f}_R^{\text{NQ}}\|_\infty \leq C \cdot 2^{-\frac{1}{\lambda}R}. \quad (4.3)$$

Thus  $D(R) = O(2^{-R/\lambda})$  as  $R \rightarrow \infty$  and  $R(D) = O(\log_2(1/D))$  as  $D \rightarrow 0$ . This shows that maximum error can be made to decay to zero at least exponentially fast, with an exponent  $\lambda^{-1}$ , as the bits per spatial Nyquist-interval goes to infinity. The constant  $C$  depends on the choice of interpolation kernels. We now argue, informally, that the rate of decay of the maximum error cannot be much better than exponential in the bitrate. Every sequence  $\{z_l\}_{l \in \mathbb{Z}}$  of values for the samples at the Nyquist-locations, essentially determines an amplitude limited  $f$  in  $BL([- \pi\lambda, \pi\lambda])$  via (4.2), with  $f(l/\lambda) \approx z_l$  for all  $l \in \mathbb{Z}$ . Let  $\{z_l\}_{l \in \mathbb{Z}}$  be a sequence whose quantization error magnitudes  $|z_l - Q_k(z_l)|$ ,  $l \in \mathbb{Z}$ , are close to  $2^{-k}$  and whose signs are such that  $\lim_{L \rightarrow \infty} \sum_{l=-L}^L (z_l - Q_k(z_l)) \phi_\lambda(t - l/\lambda)$  is close to  $C 2^{-k}$ . For such an  $f$  we have  $D(R) \approx C \cdot 2^{-\frac{1}{\lambda}R}$ . Thus informally,

$$\sup_f \|f - \hat{f}_R^{\text{NQ}}\|_\infty = O \left( 2^{-\frac{1}{\lambda}R} \right).$$

That is, for Nyquist-sampling with uniform scalar quantization and a given choice of interpolation kernels, the maximum reconstruction error for the worst amplitude limited field in  $BL([- \pi, \pi])$  decays exponentially in the bitrate. A more precise statement of this result was formally established in [20, Sec. II.B] by computing the Kolmogorov  $\epsilon$ -entropy for the class of bounded square-integrable bandlimited functions.

### B. Stochastic case

Let  $X$  be as in Section II-A.2 (also see normalization assumptions in the last paragraph of Section III). Then in view of Fact 4.1, Lemma 3.1 with  $t_l = l\lambda^{-1}$  and  $\psi_l(t) = \phi_\lambda(t)$ ,  $(l, t) \in \mathbb{Z} \times \mathbb{R}$ , and Fact 3.1, almost-surely for all sample paths of  $X$ , we have  $\forall t \in \mathbb{R}$ ,

$$X(t) = \lim_{L \rightarrow \infty} X_L(t), \quad X_L(t) = \sum_{l=-L}^L X(l/\lambda) \phi_\lambda \left( t - \frac{l}{\lambda} \right), \quad (4.4)$$

where the above series converges absolutely for each  $t$  and uniformly on all compact subsets of  $\mathbb{R}$ . In fact, for each  $t \in \mathbb{R}$ , the sequence of partial sums  $X_L$  in (4.4), which are random variables, converges to  $X(t)$  also in the expected  $p$ -th power sense,  $p \in (0, \infty)$ , due to the following fact.

**Fact 4.2:** (*From almost-sure convergence to expected  $p$ -th power convergence*) If as  $L \rightarrow \infty$ , almost-surely  $|X_L - X| \rightarrow 0$ , and  $|X_L| \leq A < \infty$  for all  $L$  (then automatically  $|X| \leq A$



almost-surely), then by the bounded convergence theorem for random variables [21, p. 17],  $\lim_{L \rightarrow \infty} \mathbb{E}|X_L - X|^p \rightarrow 0$  for all  $p \in (0, \infty)$ .

For each  $t$ , let the bitrate- $R$  quantized Nyquist-reconstruction be defined as:

$$\hat{X}_R^{\text{NQ}}(t) := \lim_{L \rightarrow \infty} \sum_{l=-L}^L Q_k(X(l/\lambda)) \phi\left(t - \frac{l}{\lambda}\right),$$

where for the same reasons as for (4.4), almost-surely and in the expected  $p$ -th power sense,  $p \in (0, \infty)$ , the above series converges absolutely for each  $t$  and uniformly on all compact subsets of  $\mathbb{R}$ . Thus as in (4.3) we have, almost-surely,

$$D(R) = \|X - \hat{X}_R^{\text{NQ}}\|_{\infty} \leq C 2^{-\frac{1}{\lambda} R}. \quad (4.5)$$

This also implies that

$$\|(\mathbb{E}|X - \hat{X}_R^{\text{NQ}}|^p)^{1/p}\|_{\infty} \leq C 2^{-\frac{1}{\lambda} R},$$

for all  $p \in (0, \infty)$ . As discussed in Section II-B, the bound  $C 2^{-\frac{1}{\lambda} R}$  in (4.3) and (4.5) also applies to the locally-averaged reconstruction errors. Since  $R \propto k$  in the Nyquist-sampling framework, the reconstruction quality can be improved only by using higher precision ADCs.

## V. ONE-BIT DITHERED OVERSAMPLING (IMPROVING QUALITY THROUGH SENSOR DENSITY)

In the Nyquist-sampling framework of the last section, samples are collected at regular Nyquist-intervals and the “bit-budget” for each interval is exhausted at a single sampling location. In this framework, the reconstruction accuracy can be improved only by improving the precision of the ADCs of the sensors. High-precision ADCs are expensive and have higher power requirements. Future sensor networks are envisioned to be made of cheap, low-power, low-precision devices deployed in large numbers over a given geographical area of interest. The question which arises naturally in this context is whether it is possible to realize high-resolution field reconstruction using a dense network of low-precision sensors. Can the fixed sensor-precision be adequately compensated through high sensor-density? If so, how does the reconstruction quality scale with the sensor density? In particular, can the maximum reconstruction error over space be made to decay exponentially with the bit-budget per Nyquist-interval as in the Nyquist-sampling framework? The answers to these questions will help characterize the “information density” in bits per meter associated with sampling a bandlimited field independent of sensor-precision. The focus of this section is on high-resolution field reconstruction using one-bit-precision sensors in the deterministic and stochastic cases. We first summarize the results for the deterministic case due to Cvetković, and Daubechies from [11] and then develop the stochastic counterpart. In the next section we study the general case of field reconstruction with  $b$ -bit-precision sensors in the deterministic and stochastic cases and expose an underlying “conservation of bits” principle.

There is a vast body of literature available on classical one-bit oversampled analog-to-digital conversion. This includes (i) oversampled sigma-delta modulation used in audio consumer

equipment for many years [3]–[5], [22], and (ii) oversampled randomized dithered averaging [6], [23]–[26] which has connections with stochastic resonance theory in physics, nonlinear dynamical systems, neural sensory information processing [27]–[29], and has been used for real-time synthetic-aperture-radar imaging [30], [31] (also see and references therein). One limitation of methods like sigma-delta modulation is that they are not amenable to distributed implementation because sampling is done in a sequential manner where the result of quantization at the previous sampling location needs to be available at the next location for quantization. A second limitation is that in all these methods, the bitrate is proportional to the oversampling rate. As a result, they have a *polynomial* distortion-rate decay characteristic, example,  $D(R) = O(R^{-(m+1)})$  for  $m$ -stage oversampled sigma-delta conversion [3], [4], [22] where  $D(R)$  is the maximum reconstruction error and  $R$  is the bit-budget per Nyquist-interval. Hence, there is a fundamental *exponential performance gap* between classical methods and Nyquist-sampling (compare with (4.3)).

A key idea for achieving an exponential distortion-rate decay characteristic, as in (4.3), using fixed-precision ADCs, is dithered oversampling. This idea was first proposed by Cvetković and Daubechies in [11] for the deterministic case using single-bit ADCs. The details of this idea and our extensions to the stochastic case are described in the following subsections. Dithered oversampling with multi-bit ADCs is discussed in the next section.

### A. Deterministic case

Let  $f$  be any unit amplitude limited field in  $BL([- \pi, \pi])$ . The key component of dither-based sampling schemes is the dither field  $d(t)$  that has the following properties:

- 1)  $\forall l \in \mathbb{Z}, \quad 1 < \gamma := |d(l/\lambda)| < \infty$ ;
- 2)  $\forall l \in \mathbb{Z}, \quad \text{sign}[d(l/\lambda)] = -\text{sign}[d((l+1)/\lambda)]$ ;
- 3)  $d(t)$  is continuous everywhere and differentiable everywhere except possibly in  $\{(l/\lambda)\}_{l \in \mathbb{Z}}$  where it is left and right differentiable. All differentials are uniformly bounded in magnitude by  $\Delta \in (0, \infty)$ .

For example,  $d(t) = \gamma \cos(\lambda \pi t)$ ,  $t \in \mathbb{R}$ , with  $|\gamma| > 1$  is a valid dither field. Sensors “add” the value of a pre-designed smooth dither field  $d(t)$ ,  $t \in \mathbb{R}$ , at their respective locations to the value of their observations  $f(t)$  and note only the *sign* of the sum. In terms of hardware, this can be implemented using threshold comparators based on operational amplifiers. Since  $f$  is continuous, the dithered field  $[f + d]$  is continuous. The first two properties of  $d$  and the unit amplitude limit on  $f$  guarantee that  $[f + d](l/\lambda)$  and  $[f + d]((l+1)/\lambda)$  will have opposite signs. By the intermediate value theorem for continuous functions [32], it follows that  $[f + d]$  will have a zero-crossing (at least one but possibly more) in every open Nyquist-interval  $(l/\lambda, (l+1)/\lambda)$ .

Let  $N = 2^k$  ( $k \geq 1$ ), one-bit ADCs be placed uniformly in every Nyquist-interval to record the sign of the dithered field  $[f + d]$ , that is, sensors are placed at the locations  $\tau_k \mathbb{Z}$  where  $\tau_k := (1/\lambda) 2^{-k}$  is the uniform oversampling period. Typically  $\tau \ll 1/\lambda$ . To avoid clutter, we shall henceforth drop the subscript  $k$  in  $\tau_k$ . Let  $m_l \in \{0, \dots, (N-1)\}$  be

the *smallest* index for which  $[f + d]((l/\lambda) + m_l\tau)$  and  $[f + d]((l/\lambda) + (m_l + 1)\tau)$  have opposite signs in  $(l/\lambda, (l+1)/\lambda)$ . It follows from the intermediate value theorem that for each  $l \in \mathbb{Z}$ ,  $f(z_l) + d(z_l) = 0$  at some point  $z_l \in ((l/\lambda) + m_l\tau, (l/\lambda) + (m_l + 1)\tau)$ . To avoid ambiguity, let  $z_l$  be the location of the *first* (from left to right) zero-crossing of  $[f + d]$  in the interval  $[l/\lambda, (l+1)/\lambda]$ . The structure of one-bit dithered sampling is depicted in Figure 3. Uniformly amplitude lim-

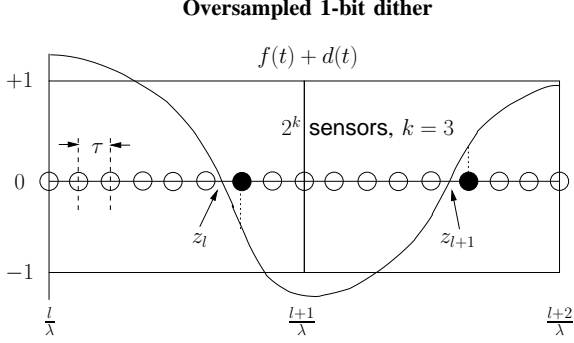


Fig. 3. **One-bit dithered sampling with 8 one-bit sensors separated by  $\tau = 1/(8\lambda)$ .** The locations of the first zero-crossing (from the left) in each Nyquist-interval are denoted by  $z_l$  in the figure. The spatially maximum interpolated reconstruction error decays exponentially in  $k$  with exponent  $(1/\lambda)$  as in the Nyquist sampling framework.

ited and bandlimited fields have uniformly amplitude limited derivatives. Specifically, according to Bernstein's inequality (2.1) in Section II-A, for unit amplitude limited fields in  $BL([- \pi, \pi])$ ,  $\forall t \in \mathbb{R}$ ,  $|f'(t)| \leq \pi$ . The third condition on the dither field and (2.1) together ensure that the value of the derivative of  $[f + d]$  is uniformly bounded by  $(\pi + \Delta)$ . Applying Lagrange's mean value theorem [32] to the mid-point of the "uncertainty zone"  $t_l := ((l/\lambda) + (m_l + (1/2))\tau)$  and  $z_l$ , we get

$$|[f + d](z_l) - [f + d](t_l)| \leq (\pi + \Delta) |z_l - t_l|, \quad \text{that is,}$$

$$|f(t_l) - (-d(t_l))| \leq \left(\frac{\pi + \Delta}{2}\right) \tau = \left(\frac{\pi + \Delta}{2\lambda}\right) 2^{-k}.$$

Thus, uniform oversampling of the dithered field using one-bit ADCs gives samples of  $f$  having linear precision in  $\tau$  (exponential in  $k$ ) at the *nonuniformly* spaced locations  $\mathcal{T} = \{t_l\}_{l \in \mathbb{Z}}$  which depend on both the field and the dither. Loosely speaking, the dither field provides a way of translating spatial resolution/uncertainty of the zero-crossing locations (but amplitude certainty, equal to zero) to amplitude resolution/uncertainty of the field at these locations (but spatial certainty, equal to the mid-point of the uncertainty zone).

Unlike Nyquist-sampling where the bit-budget for each Nyquist-interval is exhausted at a single sampling point, the dither-based approach "spreads" the bit-budget over many single-bit ADCs in a Nyquist-interval. There are  $N = 2^k$  one-bit sensors uniformly distributed over an interval of length  $(1/\lambda)$ . It takes  $k$  bits or a bitrate of  $R = k\lambda = O(\log N)$  bits per Nyquist-interval per field snapshot to index the location of the first zero-crossing. Deferring a full discussion of distributed processing and communication issues and costs to Section VII, as a conceptual aid one may imagine that when a

sensor observes a sign change relative to its neighbors, it can wirelessly broadcast its identity to the CPU using  $\log N$  bits with all other sensors remaining silent. For each  $l \in \mathbb{Z}$ , the sample errors decay with rate  $R$  as follows

$$|f(t_l) - (-d(t_l))| \leq \left(\frac{\pi + \Delta}{2\lambda}\right) 2^{-\frac{1}{\lambda}R}. \quad (5.1)$$

This per-sample amplitude resolution, which is proportional to inter-sensor separation, can be converted to global field amplitude resolution of the same order using the following result on field interpolation from nonuniformly spaced samples (adapted from [11, Theorem 3.2]) due to Cvetković and Daubechies.

*Fact 5.1: (Stable interpolation of bandlimited fields from samples at nonuniformly spaced locations [11, Theorem 3.2])* Let  $\lambda > 1$  and  $\mathcal{T} = \{t_l\}_{l \in \mathbb{Z}}$  be a set of sampling locations such that  $\underline{\kappa} := \inf_{j, l \in \mathbb{Z}, j \neq l} |t_j - t_l| > 0$  and  $\bar{\kappa} := \sup_{l \in \mathbb{Z}} |t_l - (l/\lambda)| < \infty$ . Then there exist absolutely-integrable interpolation kernels  $\psi_l$  belonging to  $BL([- \pi\lambda, \pi\lambda])$  such that  $C' := \sup_{t \in \mathbb{R}} [\sum_{l \in \mathbb{Z}} |\psi_l(t - t_l)|] < \infty$  and for all fields  $f \in BL([- \pi\lambda + \delta, \pi\lambda - \delta])$  with  $0 < \delta < \pi\lambda$ ,

$$f(t) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L f(t_l) \psi_l(t - t_l),$$

where the above series converges absolutely for each  $t$  (and in  $\mathcal{L}^2$ ) and uniformly on all compact subsets of  $\mathbb{R}$ . The interpolation kernels  $\psi_l$  depend on the sampling set  $\mathcal{T}$  but there exist interpolation kernels for which  $C'$  depends only on  $\lambda$ ,  $\underline{\kappa}$ , and  $\bar{\kappa}$  and not on the specific sampling set  $\mathcal{T}$ .

In essence, the field can be reconstructed in a stable manner as in the Nyquist-sampling framework as long as the sampling locations do not get too close to each other (not closer than  $\underline{\kappa} > 0$ ) and do not run away too far from the Nyquist-points (not farther than  $0 < \bar{\kappa} < \infty$ ). There exist interpolation kernels  $\psi_l$  that decay faster than  $(c'_m/t^m)$  for all  $m$  where the constants  $\{c'_m\}_{m \in \mathbb{N}}$  depend only on  $\lambda$ ,  $\underline{\kappa}$ , and  $\bar{\kappa}$  and not on  $\mathcal{T}$  [11]. As discussed in Section VII-B.2, these properties prove useful in the practical context of reconstructing the bandlimited field to a desired *nonzero* distortion from samples collected in only a *finite* spatial region of interest.

In Remark 5.1 below, it is shown that  $\mathcal{T} = \{t_l := ((l/\lambda) + (m_l + (1/2))\tau)\}_{l \in \mathbb{Z}}$  satisfies the conditions of Fact 5.1 with  $\underline{\kappa}$ ,  $\bar{\kappa}$ , and  $C'$  which do not depend on  $\tau$ . Hence, from (5.1), Fact 5.1, and the triangle inequality, it follows that there exist interpolation kernels  $\{\psi_l\}_{l \in \mathbb{Z}}$  (which depend on  $\mathcal{T}$  in general) for which the spatially maximum interpolated reconstruction error is bounded as follows (also see [11, Corollary 3.3])

$$D(R) := \|f - \hat{f}_R^{\text{one-bit}}\|_\infty \leq C' \left(\frac{\pi + \Delta}{2\lambda}\right) 2^{-\frac{1}{\lambda}R}, \quad (5.2)$$

$$\text{where } \hat{f}_R^{\text{one-bit}}(t) := \sum_{l \in \mathbb{Z}} (-d(t_l)) \psi_l(t - t_l).$$

The reconstruction error decays exponentially with the bitrate, with the same exponent as in the Nyquist-sampling scheme (4.3). The reconstruction accuracy can be improved by reducing  $\tau$ , that is, by packing more sensors inside each Nyquist-interval. However, unlike the Nyquist-sampling scheme, there is no need to use higher precision ADCs.



*Remark 5.1:* We are interested in the behavior of the reconstruction error as the number of sensors  $N = 2^k = 2^{R/\lambda}$  increases. Since  $C'$  depends on  $\underline{\kappa} = \inf_{j \neq l} |t_j - t_l|$  and  $\bar{\kappa} := \sup_{l \in \mathbb{Z}} |t_l - (l/\lambda)|$  (but the interpolation kernels  $\psi_l$ s can be designed so that  $C'$  does not depend on the specific nonuniform sampling set), to truly assert that the decay of the reconstruction error is exponential in  $R$ , we must ensure that  $\underline{\kappa}$  and  $\bar{\kappa}$  do not depend on  $\tau = (1/\lambda)2^{-k}$ . Clearly,  $\bar{\kappa} < 1/\lambda$ . What about  $\underline{\kappa}$ ? A lower bound on  $\underline{\kappa}$  is  $\tau$  since the  $\{t_l\}_{l \in \mathbb{Z}}$  cannot get closer than  $\tau/2$  to the endpoints of the Nyquist-interval by design. This lower bound depends on  $\tau$ . However, since  $|d(t) + f(t)| \geq |d(t)| - |f(t)| \geq |d(t)| - 1$ , a zero-crossing cannot occur unless the amplitude of  $d$  falls below unity. Since  $|d(l/\lambda)| = \gamma > 1$ , the closest that a zero-crossing can get to a Nyquist-point is  $(\gamma - 1)/\Delta$  where  $\Delta$  is the maximum slope of  $d$ . Since the  $t_l$ s are midpoints of inter-sensor intervals,  $\underline{\kappa} \geq (2(\gamma - 1)/\Delta) - \tau$ . Together with the earlier lower bound  $\underline{\kappa} \geq \tau$ , this means that  $\underline{\kappa} \geq (\gamma - 1)/\Delta$  which is independent of  $\tau$ . For aiding comparison with  $b$ -bit dithered oversampling of the next section, we take  $C'$  to correspond to the conservative values  $\bar{\kappa}^{\text{one-bit}} = 1/\lambda$  and  $\underline{\kappa}^{\text{one-bit}} = \min(1/(2\lambda), (\gamma - 1)/\Delta)$ .

*Remark 5.2:* From (5.2), we would like the maximum slope  $\Delta$  of the dither field to be as small as possible to get a tighter upper bound on the error. The average slope of the dither field in any Nyquist-interval is  $(2\gamma\lambda)$ . Since  $d$  is smooth on every open Nyquist-interval, by the mean-value theorem, there is always some point in the open interval where the slope of  $d$  matches the average slope. Hence,  $\Delta$  can be no smaller than  $(2\gamma\lambda)$ . We can make  $\Delta = (2\gamma\lambda)$  by using a piece-wise linear dither field.

*Remark 5.3:* It is in fact possible to design a dither field  $d$  for which  $[f + d]$  will have *exactly one* zero-crossing in each Nyquist-interval: consider for example the continuous periodic triangular dither field which is piecewise linear in each Nyquist-interval and whose slope  $\Delta$  is strictly greater than  $\pi$  in magnitude. For such a  $d$ , the sum field  $[f + d]$  is strictly monotonic in each Nyquist-interval because the slope of the sum field never changes sign in any Nyquist interval. For expositional simplicity and to aid the discussion of communication issues and costs in Section VII, for the remainder of this paper, we will assume that the dither field satisfies this property.

## B. Stochastic case

Let  $X$  be as in Section II-A.2 (also see normalization assumptions in the last paragraph of Section III). It should be noted that  $X$  is a random process and that its sample paths are not bandlimited in the conventional sense. There may even be discontinuous sample paths (having probability zero) for which  $[X + d]$  may not have an actual zero-crossing. However, since  $|X| \leq 1$  and  $\gamma > 1$ ,  $[X + d](l/\lambda)$  and  $[X + d]((l+1)/\lambda)$  will always have opposite signs. Hence, in each Nyquist-interval, there always exists a first sensor which will detect a *sign-change*. As in the deterministic case, let  $M_l$  be the index (a discrete random variable) of the sensor located just left of the first sign-change and  $T_l := [(l/\lambda) + (M_l + (1/2))\tau]$ . Note that the set  $\mathcal{T} = \{T_l\}_{l \in \mathbb{Z}}$  is a countable set of random “zero-crossings” (more accurately, first sign-change locations) which

is closely coupled to the underlying spatial stochastic process  $X$ . However, each  $T_l$  can take values only in the deterministic discrete set  $\{(l/\lambda) + (m + 0.5)\tau : m = 0, \dots, (N - 1)\}$ .

From Fact 3.1 and the Remark 5.1, almost-surely the random sampling set  $\mathcal{T}$  satisfies the conditions of Fact 5.1. Then from Fact 5.1 and Lemma 3.1 with  $C = C'$ , almost-surely for all sample paths of  $X$ ,  $\forall t \in \mathbb{R}$ ,

$$X(t) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L X(T_l) \psi_l(t - T_l), \quad (5.3)$$

where the above series converges absolutely for each  $t$  and uniformly on all compact subsets of  $\mathbb{R}$ . In fact, due to Fact 4.2, for each  $t \in \mathbb{R}$ , the sequence of partial sums to the right of (5.3), which are random variables, converges to  $X(t)$  also in the expected  $p$ -th power sense,  $p \in (0, \infty)$ .

From Fact 3.1 and Proposition 3.1, almost-surely, all sample paths of  $X$  are differentiable everywhere, and the derivative  $X'$  has a normalized amplitude which is uniformly bounded by  $\pi$ . Hence, by (5.1), almost-surely for all paths of  $X$ ,

$$|X(T_l) - (-d(T_l))| \leq \left( \frac{\pi + \Delta}{2\lambda} \right) 2^{-\frac{1}{\lambda}R}, \quad (5.4)$$

$\forall l \in \mathbb{Z}$ . For each  $t \in \mathbb{R}$ , let the bitrate- $R$ , one-bit quantized reconstruction of  $X$  be defined as:

$$\hat{X}_R^{\text{one-bit}}(t) := \sum_{l \in \mathbb{Z}} (-d(T_l)) \psi_l(t - T_l), \quad (5.5)$$

where for the same reasons as for (5.3), almost-surely and in the expected  $p$ -th power sense,  $p \in (0, \infty)$ , the above series converges absolutely for each  $t$  and uniformly on all compact subsets of  $\mathbb{R}$ . Thus as in (5.2), almost-surely,

$$D(R) = \|X - \hat{X}_R^{\text{one-bit}}\|_\infty \leq C'' 2^{-\frac{1}{\lambda}R}, \quad (5.6)$$

with  $C'' := C'(\pi + \Delta)/(2\lambda)$ . This also implies that

$$\|(\mathbb{E}|X - \hat{X}_R^{\text{one-bit}}|^p)^{1/p}\|_\infty \leq C'' 2^{-\frac{1}{\lambda}R},$$

for all  $p \in (0, \infty)$ . Once again, as discussed in Section II-B, the bound  $C'' 2^{-\frac{1}{\lambda}R}$  in (5.2) and (5.6) also applies to the locally-averaged reconstruction errors. It should be noted that  $C'$  depends only on  $\lambda$ ,  $\underline{\kappa}$ , and  $\bar{\kappa}$ , which are fixed, and not on the specific realization of the set of random sampling locations  $\mathcal{T}$  which satisfy the assumptions of Fact 5.1. However, in general the set of interpolation kernels  $\{\psi_l\}_{l \in \mathbb{Z}}$  depends on the *random* sampling set  $\mathcal{T}$ .

*Discussion:* In the stochastic one-bit dithered oversampling scenario of this and the following section, the sampling locations  $\mathcal{T}$  and the associated interpolation kernels are tightly coupled to the underlying process  $X$  because they arise as the locations of level-crossings of the process  $X$ . Most results in the nonuniform sampling literature (see [6], [25], [26] and references therein) address the reconstruction of (i) deterministic bandlimited fields from nonuniform deterministic sampling locations, (ii) bandlimited stochastic processes from nonuniform deterministic sampling locations, (iii) deterministic bandlimited fields from random sampling locations, and (iv) bandlimited stochastic processes from *independent* random sampling locations. Scenarios (iii) and (iv) may arise, for

example, due to random sensor-deployment independent of the field being sensed. However, the results here are for the reconstruction of WSS bandlimited stochastic processes from random sampling locations which are *strongly dependent* on the process being sampled.

Summarizing (4.3), (4.5), (5.2), and (5.6), it is clear that for a fixed  $\lambda, \gamma > 1$ , the  $\mathcal{L}^\infty$  norm of the point-wise error in the Nyquist-reconstruction or one-bit dithered reconstruction decays at least exponentially fast in the bitrate (bits per spatial Nyquist-interval for each field snapshot)  $R$  with exponent  $(1/\lambda)$ . Similar rate-error characteristics hold for the deterministic and expected local-average distortion criteria of Section II. Hence, if  $D$  denotes the maximum point error, the “intrinsic information density” (ignoring data-transport issues)  $R(D)$  in bits per meter grows no faster than (with the almost-sure/expected qualifiers in the stochastic case)  $R(D) = O(\log(1/D))$  as  $D$  decreases to 0. For one-bit dithered sampling, the number of sensors per Nyquist-interval  $N$ , grows no faster than  $O(1/D)$ , or equivalently,  $D(N) = O(1/N)$ . The total information in bits per Nyquist-interval grows no faster than  $O(\log N)$  and the bits per sensor decreases at least as fast as  $O((1/N) \log N)$ .

## VI. TRADEOFFS IN AMPLITUDE AND SPATIAL RESOLUTION: A “BIT-CONSERVATION” PRINCIPLE

In Nyquist-sampling, fields are sampled at “low”, near-Nyquist-rates. The entire bit-budget of  $k$  bits per Nyquist-interval is spent in recording the field amplitude at a single high-precision ( $k$ -bit) sensor. On the other hand, the one-bit dither-based sampling scheme spends all available bits in recording the location of a zero-crossing (a spatial event) by using many ( $N = 2^k$ ) poor precision (one-bit) sensors in each Nyquist-interval. These sampling schemes represent two extreme scenarios. This section explains how  $k$ -bit Nyquist-sampling accuracy can be achieved using  $b$ -bit ADCs and an appropriate dither-based oversampling scheme for any  $1 < b < k$ . This leads to a bit-conservation principle – a tradeoff between the oversampling factor and ADC-precision for “similar” asymptotic reconstruction accuracy. To the best of our knowledge, this tradeoff has not been discussed before in the sampling literature. We discuss this in detail for deterministic fields. The extension to the stochastic setting parallels the stochastic extensions in Sections IV-B and V-B.

### A. Deterministic case

Nyquist-sampling uses only one  $k$ -bit ADC per Nyquist-interval of length  $1/\lambda$ . The one-bit dithered sampling scheme uses  $N = 2^k$ , one-bit ADCs distributed uniformly over the same interval, that is, the ADCs are placed at intervals of length  $\tau = 1/(N\lambda)$ . For definiteness, assume that the sensors are placed at the beginning of every  $\tau$ -length interval, that is, at locations  $\{m\tau\}_{m \in \mathbb{Z}}$ . Now consider the scenario where  $N_b := 2^{k-b+1}$   $b$ -bit ADCs ( $1 < b < k$ ) are placed  $\tau$  apart, in the leftmost section of the Nyquist interval. Notice that, one-bit ADCs only detect one level-crossing (the 0 level), 2-bit ADCs can detect 3 distinct level-crossings, and in general,  $b$ -bit ADCs can detect  $(2^b - 1)$  distinct level-crossings given by  $\{0, \pm(1/2^{b-1}), \dots, \pm(1 - (1/2^{b-1}))\}$ .

Appendix IV shows how to design a  $b$ -bit dither field  $d_b$ , with maximum slope magnitude  $\tilde{\Delta}$  independent of  $b$ , so that  $[f + d_b]$  always crosses some quantization level in every interval of the form  $[A_l, B_l] := [l/\lambda, (l + (N_b - 1)/N)/\lambda] \subset [l/\lambda, (l + 1)/\lambda]$ .<sup>9</sup> Each interval  $[A_l, B_l]$  usually covers only a small fraction of the length of a Nyquist-interval and contains  $N_b$   $b$ -bit ADCs. The situation is graphically illustrated in Figure 4.

Let  $m_l \in \{0, \dots, (N_b - 2)\}$  be the smallest index for which  $[f + d_b](A_l + m_l\tau)$  and  $[f + d_b](A_l + (m_l + 1)\tau)$  are on opposite sides of some level  $q_l \in \{0, \pm(1/2^{b-1}), \dots, \pm(1 - (1/2^{b-1}))\}$  and let  $z_l \in (A_l + m_l\tau, A_l + (m_l + 1)\tau)$  be the actual point of level-crossing, that is,  $f(z_l) + d_b(z_l) = q_l$ . From the mean value theorem applied to  $[f + d_b](t)$  for the end-points  $t_l := A_l + (m_l + (1/2))\tau$  and  $z_l$  we obtain

$$|[f + d_b](z_l) - [f + d_b](t_l)| \leq \sup_{t \in (A_l, B_l)} |[f' + d'_b](t)| |z_l - t_l|,$$

which leads to,

$$|f(t_l) - (q_l - d_b(t_l))| \leq \left( \frac{\pi + \tilde{\Delta}}{2} \right) \tau.$$

This shows that the accuracy of the nonuniform samples  $\{f(t_l)\}$  is linear in  $\tau$ , independent of the precision of the ADCs just as in the one-bit dithered sampling scheme.

It requires only  $\lceil \log_2(N_b - 1) \rceil \leq (k - b + 1)$  bits per Nyquist-interval to specify the location of the sensor *just* following the location of the first level-crossing of  $[f + d_b](t)$ . It takes no more than  $\log_2(2^b - 1) < b$  bits to index the level that was crossed first in each Nyquist-interval. Hence, the total number of bits required with this sampling method is not more than  $(k - b + 1) + b = (k + 1)$  bits (or  $R = \lambda(k + 1)$  bits per interval) which gives the same distortion-rate decay characteristics as the  $k$ -bit Nyquist-sampling and the one-bit dither-based sampling schemes. As in Section V, we defer a full discussion of communication issues and costs to Section VII. As a conceptual aid, however, one may imagine that the sensor observing a level-crossing relative to its neighbor wirelessly broadcasts its index and the index of the level that has been crossed to the CPU using  $(k + 1)$  bits with all other sensors remaining silent. The source of this additional bit can be explained as follows: In one-bit dithered sampling, the ADCs do not need to explicitly specify which level was crossed since there is only one level; only zero-crossing locations need to be described. In the  $k$ -bit Nyquist-sampling, the locations of the ADCs need not be explicitly specified since they are known; however, the sensors need to specify the quantization interval in which the field sample lies.

Since  $\tau$  and  $k$  are related via  $\tau = 1/(\lambda 2^k)$ , therefore,  $\tau = (2/\lambda)2^{-\frac{1}{\lambda}R}$ , and the maximum sample error is no more than

$$|f(t_l) - (q_l - d_b(t_l))| \leq \left( \frac{\pi + \tilde{\Delta}}{\lambda} \right) 2^{-\frac{1}{\lambda}R}, \quad (6.1)$$

<sup>9</sup>In fact, it is possible to construct  $d_b(t)$  using any dither field  $d(t) = d_1(t)$  of the one-bit sampling scheme via Equation (IV.1) in Appendix IV. Then, the maximum slope of the  $b$ -bit dither field in any interval  $(A_l, B_l)$  is no more than  $\tilde{\Delta} = (4c/\gamma)\Delta$ ,  $c > (1 + \pi)/4$ , where  $\Delta$  is the maximum slope of the one-bit dither in any interval  $(l/\lambda, (l + 1)/\lambda)$ .

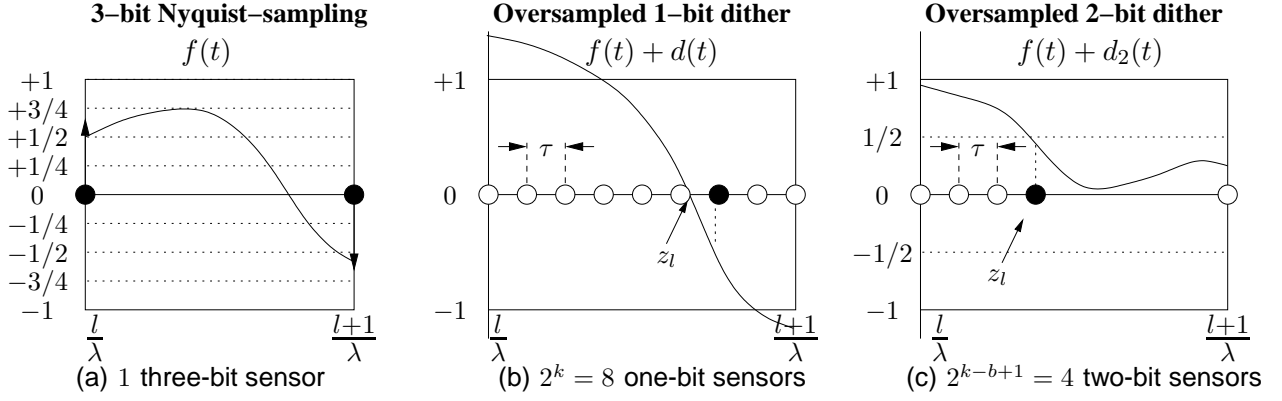


Fig. 4. **Amplitude resolution versus oversampling rate tradeoffs in conventional and dither-based sampling frameworks.** (a) Conventional (Nyquist) sampling using 3-bit ADCs placed at Nyquist-locations  $\{l/\lambda\}_{l \in \mathbb{Z}}$ . The entire budget of 3 bits is exhausted at a single sample point in any Nyquist-interval. (b) Dither-based sampling scheme using 8, one-bit ADCs distributed uniformly over a Nyquist-interval to locate the zero-crossing  $z_l$  of the dithered field. (c) A flexible tradeoff between these extremes. Four, 2-bit ADCs distributed uniformly over half the Nyquist-interval detect a level-crossing  $z_l$  at level  $+(1/2)$ . All three schemes have similar exponential error accuracy in the bitrate.

$\forall l \in \mathbb{Z}$ . Following the analysis for the one-bit case, it can be verified that  $\{t_l\}_{l \in \mathbb{Z}}$  satisfies the conditions of Fact 5.1. Specifically,  $\bar{\kappa} \leq (B_l - A_l) = 1/(\lambda 2^{(b-1)}) \leq 1/\lambda = \bar{\kappa}^{\text{one-bit}}$  and  $\underline{\kappa} \geq (A_{l+1} - B_l) \geq (1 - (1/2^{(b-1)}))(1/\lambda) \geq 1/(2\lambda) \geq \underline{\kappa}^{\text{one-bit}}$  (see Remark 5.1). Hence, if the rate- $R$ ,  $b$ -bit reconstruction is given by  $\hat{f}_R^{b\text{-bit}}(t) := \sum_{l \in \mathbb{Z}} (q_l - d_b(t_l)) \psi_l(t - t_l)$ ,

$$\|f - \hat{f}_R^{b\text{-bit}}\|_\infty \leq \tilde{C}' \left( \frac{\pi + \tilde{\Delta}}{\lambda} \right) 2^{-\frac{1}{\lambda} R}, \quad (6.2)$$

where the constant  $\tilde{C}'$  (tilde is used to distinguish the constants of this section from those in Section V) does not depend on  $\tau$ ,  $f$ , and the individual  $\{t_l\}_{l \in \mathbb{Z}}$  but only on  $\underline{\kappa}$ ,  $\bar{\kappa}$ , and  $\lambda$ . If we use the conservative values  $\bar{\kappa} = \bar{\kappa}^{\text{one-bit}}$  and  $\underline{\kappa} = \underline{\kappa}^{\text{one-bit}}$  then  $\tilde{C}' = C'$ . Hence, independent of  $b$ , the distortion-rate asymptotics are the same as in the Nyquist-sampling and the one-bit dithered sampling set-ups, that is, the distortion is exponentially decaying in rate  $R$  with the same exponent. This observation is summarized in the following principle.

**“Conservation of bits” principle:** Let  $k$  be the bit-budget in terms of the total number of bits available per Nyquist-interval. For each  $1 \leq b < k$  there exists a (dither-based) sampling scheme with not more than  $2^{k-b+1}$ ,  $b$ -bit sensors per Nyquist-interval achieving a spatially maximum reconstruction error  $D$  of the order of  $2^{-k}$ . Alternatively, if there are  $N_b$ ,  $b$ -bit sensors in each Nyquist-interval, then  $D = O(2^{-(b+\log_2 N_b)})$ . The maximum error decays exponentially in the bit-budget. The bit-budget can be apportioned in a flexible manner between amplitude resolution in bits  $b$  and the spatial resolution in bits  $\log_2 N_b$ . It should be noted that  $N_b$  is the sensor density in terms of the number of sensors per Nyquist-interval.

### B. Stochastic case

The procedure for extending the results from the deterministic to the stochastic  $b$ -bit setting is identical to the stochastic extension of

the one-bit deterministic dither scheme.  $X$  is sampled by adding  $d_b$  having the same properties as in the deterministic case. Let  $M_l$  be the index of the sensor just prior to the first level-change,  $T_l := [l/\lambda + (M_l + (1/2))\tau]$  the mid-point of the interval in which the first level change is detected, and  $q_l$  be the level crossed.

From Fact 3.1 and Remark 5.1, almost-surely the random sampling set  $\mathcal{T}$  satisfies the conditions of Fact 5.1. In addition, from Fact 3.1 and Proposition 3.1, almost-surely, all sample paths of  $X$  are differentiable everywhere, and the derivative  $X'$  has a normalized amplitude which is uniformly bounded by  $\pi$ . Hence, almost-surely for all paths of  $X$ , the expansion (5.3) holds, and by (6.1),

$$|X(T_l) - (q_l - d_b(T_l))| \leq \left( \frac{\pi + \tilde{\Delta}}{\lambda} \right) 2^{-\frac{1}{\lambda} R}, \quad (6.3)$$

$\forall l \in \mathbb{Z}$ . For each  $t \in \mathbb{R}$ , let the bitrate- $R$ ,  $b$ -bit quantized reconstruction of  $X$  be defined as:

$$\hat{X}_R^{b\text{-bit}}(t) := \sum_{l \in \mathbb{Z}} (q_l - d_b(T_l)) \psi_l(t - T_l), \quad (6.4)$$

where for the same reasons as for (5.3), almost-surely and in the expected  $p$ -th power sense,  $p \in (0, \infty)$ , the above series converges absolutely for each  $t$  and uniformly on all compact subsets of  $\mathbb{R}$ . Thus as in (6.2), almost-surely,

$$D(R) = \|X - \hat{X}_R^{b\text{-bit}}\|_\infty \leq \tilde{C}'' 2^{-\frac{1}{\lambda} R}, \quad (6.5)$$

with  $\tilde{C}'' := (\tilde{C}'(\pi + \tilde{\Delta})/\lambda)$ . This also implies that

$$\|(\mathbb{E}|X - \hat{X}_R^{b\text{-bit}}|^p)^{1/p}\|_\infty \leq \tilde{C}'' 2^{-\frac{1}{\lambda} R},$$

for all  $p \in (0, \infty)$ . Once again, as discussed in Section II-B, the bound  $\tilde{C}'' 2^{-\frac{1}{\lambda} R}$  in (6.2) and (6.5) also applies to the locally-averaged reconstruction errors. It should be noted that  $\tilde{C}'$  only depends on  $\lambda$ ,  $\underline{\kappa}$ , and  $\bar{\kappa}$ , which are fixed, and not on the specific realization of the set of random sampling locations  $\mathcal{T}$  which satisfy the assumptions of Fact 5.1. However, the set of interpolation kernels  $\{\psi_l\}_{l \in \mathbb{Z}}$  depends, in general, on the *random* sampling set  $\mathcal{T}$ .



### C. Discussion

**Nyquist-sampling as a special case:** Note that the Nyquist-sampling scheme is also subsumed by the proposed generalized dithered sampling framework. Indeed, for  $b = k$ , the described framework suggests using two,  $k$ -bit ADCs at locations  $(\frac{l}{\lambda})$  and  $(\frac{l}{\lambda}) + \tau$ . However, the dithered field is guaranteed to have a level-crossing in  $[\frac{l}{\lambda}, \frac{l}{\lambda} + \tau)$ . Hence, the second sensor is redundant and there is no need for the first sensor to add the dither-value.

**Number of sensors vs ADC-precision and sensor distribution:** While maintaining the same asymptotic error decay profile, for a given reconstruction quality, as the ADC-precision  $b$  increases, the number of sensors decreases exponentially with  $b$ . One-bit dithered sampling needs  $N$  sensors,  $b$ -bit ( $b > 1$ ) dithered sampling needs  $N_b = N/2^{b-1}$  sensors, and Nyquist-sampling needs only one sensor per Nyquist-interval. The number of sensors can be traded off with the precision of sensors. Although the number of sensors has been a power two in our analysis, this is only for ease of illustration and is not a restrictive assumption. Also, sensors need to be placed only in intervals of the form  $[\frac{l}{\lambda}, \frac{l}{\lambda} + (N_b - 1)\tau)$ . This leaves “inactive” regions over which the field is not sampled. Although the discussion has, thus far, focused on sampling all the Nyquist-intervals in an identical fashion, this need not be so. Some Nyquist-intervals can use fewer sensors of higher precision ( $b$  large) bunched closer together in a smaller interval. Others can use more sensors of lower precision ( $b$  small) spread over a larger interval. The key result which enables such a general non-uniform sampling using heterogeneous sensors while achieving the same exponential rate-distortion decay characteristics is Fact 5.1. This bunched irregular sampling ability provides considerable flexibility in selecting spatially adaptive sensor precision and deployment patterns. For example, in difficult terrain or in the presence of occluding obstacles, where sampling density is forced to be low, one can use high-precision sensors, and use low-precision sensors, in terrain where the sampling density can be high.

Two other side-benefits of the dither-based oversampling scheme worth mentioning (but not quantified here) include (i) a certain degree of natural robustness to node failures: if every alternate node fails, the effective inter-node separation increases to  $2\tau$  and is equivalent to halving the precision of the ADCs and (ii) some measure of data security against eavesdropping: the dither fields can be kept covert and there is considerable flexibility in their design for a given reconstruction quality.

Table I summarizes all the salient features of different sampling schemes considered in this paper.

## VII. DISTRIBUTED PROCESSING AND COMMUNICATION COSTS

Sections IV–VI focused on the quantization and interpolation aspects of sampling. In this section we focus on distributed processing and communication aspects of dithered oversampling for reconstructing a sequence of fields at a CPU in a sensor network. We first discuss the problem of

field acquisition and coding (Section VII-A). We describe two approaches for processing and coding raw sensor data, one based on local nearest-neighbor communication and the other on distributed source coding, and derive per Nyquist-interval per snapshot costs. Since spatially bandlimited fields are not spatially limited, it is impractical to move data from an infinite number of spatial Nyquist-intervals to the CPU. We address this issue by considering how the field reconstruction error in a compact region of interest decays as the field is sampled over a growing neighborhood of sensor deployment (Section VII-B). Here we focus on multihop communication to move all the processed and coded data from each spatial Nyquist-interval to the CPU. We also indicate, however, how a Gaussian multiple access uplink may be used for the same purpose.

To simplify the presentation, we focus on one-bit sensors and 1-D deterministic fields in  $BL([- \pi, \pi])$  and assume that the dither field is such that in each Nyquist-interval (i) there is exactly one zero-crossing (see Remark 5.3) and (ii) the zero-crossing direction (the sign change from say, plus to minus, moving from left to right) is fixed. In Section VII-C we briefly indicate how results can be easily extended to two and higher dimensional spatially bandlimited fields by reusing the results for 1-D fields.

### A. Field acquisition and coding

Let  $N$ , one-bit sensors be placed uniformly at every  $\tau = \frac{1}{\lambda N}$  meters in every Nyquist-interval  $[\frac{l}{\lambda}, \frac{(l+1)}{\lambda} - \tau]$ . Periodically, the sensors synchronously take snapshots of the 1-D spatially bandlimited random field by comparing the field-value to the dither-value at their respective locations. The dither-values are assumed to be pre-stored during sensor deployment. Sensor clocks are assumed to be synchronized during deployment. The temporal sequence of the signs of the field plus dither-values constitutes the *raw* data produced at each sensor.

We consider two approaches for processing and coding the raw data produced at each sensor for communication to a CPU where the field snapshots are to be reconstructed. The first approach involves (a) neighbor-to-neighbor local communication within each Nyquist-interval to locate the zero-crossings followed by (b) coding of the zero-crossing data at each sensor. The second approach does not require inter-sensor collaboration and is based on the principles of distributed source coding due to Slepian and Wolf [33], [34] (see [35] for practical code constructions). Distributed source coding exploits the *correlation* present in the sensor data, e.g., there is exactly one zero-crossing in each Nyquist-interval. Slepian and Wolf proved that the compression-efficiency of non-collaborative coding, in terms of the average rate, is equal to that of collaborative coding for discrete-valued data. The probability of correctly decoding data which has been compressed using a distributed source code tends to one as the number of snapshots (coding blocklength) approaches infinity.

1) *Coding using local communication:* (a) *Locating zero-crossings.* The sensors at the locations  $\{\frac{l}{\lambda}\}_{l \in \mathbb{Z}}$  are designated as the “leading” nodes for initiating local communication (confined to Nyquist-intervals) for determining the zero-crossing within each spatial Nyquist-interval for each time snapshot.

TABLE I  
COMPARISON OF DIFFERENT SAMPLING SCHEMES.

Sampling scheme:	Nyquist	one-bit dither	$b$ -bit dither, $1 < b < k$
Sensor-precision (bits)	$k$ -bit	one-bit	$b$ -bit
Sensors per NQ intvl.	1	$2^k$	$2^{k-b+1}$
Inter-sensor spacing	$(\frac{1}{\lambda})$	$\tau := \frac{1}{\lambda 2^k}$	$\tau := \frac{1}{\lambda 2^k}$
Sensor distribution	1 sensor at every $(\frac{1}{\lambda})$	$2^k$ sensors uniformly spaced by $\tau$ over every $[\frac{l}{\lambda}, \frac{l+1}{\lambda})$	$2^{k-b+1}$ sensors uniformly spaced by $\tau$ over every $[\frac{l}{\lambda}, \frac{l}{\lambda} + \frac{1}{\lambda 2^{b-1}})$
Bits per NQ intvl. (bits/meter)	$R = k\lambda$	$R = k\lambda$	$R = (k+1)\lambda$
<b>Distortion-Rate</b>	$D = O(2^{-\frac{1}{\lambda} \cdot R})$	$D = O(2^{-\frac{1}{\lambda} \cdot R})$	$D = O(2^{-\frac{1}{\lambda} \cdot R})$
Local comm. cost bit-meters per NQ intvl.	0	$\leq \frac{2}{\lambda}$	$\leq (\frac{b+1}{\lambda 2^{b-1}}) \leq \frac{2}{\lambda}$

Corresponding to each temporal snapshot of the field, the message passed by each node to its neighbor (the right neighbor for definiteness), indicates two things: (i) whether or not a zero-crossing has already been found by some preceding sensor and (ii) if a zero-crossing has not yet been found, the sign of the field plus dither at its location. The local communication terminates at the last (right-extreme) sensor in the Nyquist-interval. In each Nyquist-interval, the first sensor that detects a sign change between what its left neighbor reports and its own reading records a “1”. Other sensors record a “0” for that snapshot. Thus, at the *fixed* inter-sensor communication rate of 2 bits per snapshot, the raw one-bit sensor readings can be converted to zero-crossing information. Note that the local communication can be done in one shot after aggregating the data across many snapshots.

(b) *Compression of zero-crossing data.* Each sensor encodes (compresses) the zero-crossing information – the temporal sequence of zeros and ones where one indicates the presence of a zero-crossing and a zero the absence. This compression can again be done in one of two ways. The first approach, not discussed here (but see next subsection), involves using distributed source coding by exploiting the correlation in the zero-crossing information. However, an alternative to distributed source coding involves each sensor independently entropy-coding (e.g., run-length coding) its zero-crossing data ignoring the inter-sensor correlation. The compression efficiency of independent coding depends on the specific underlying correlation structure. Let  $p_i \in [0, 1], i = 1, \dots, N$ , denote the fraction of snapshots in which the zero-crossings occur at the  $i$ -th sensor in a Nyquist-interval. Since there is exactly one zero-crossing in each Nyquist-interval in each snapshot,  $\sum_{i=1}^N p_i = 1$ . Sensor  $i$  can compress its zero-crossing information to the rate of  $h_2(p_i)$  bits per snapshot, where  $h_2(p) = -p \log_2(p) - (1-p) \log_2(1-p), p \in [0, 1]$ , is the binary entropy function [34]. The total bitrate in a Nyquist-interval is given by  $\sum_{i=1}^N h_2(p_i) \leq N h_2(1/N)$  where the inequality is due to the (strict) concavity of  $h_2(\cdot)$  and Jensen’s inequality [34]. This upper bound will be attained if all sensors see the same number of zero-crossings on the average. Thus the total number of bits per Nyquist-interval per snapshot  $R_{NQ}(N)$  for independent entropy-coding is not more than  $N \cdot h_2(1/N) \leq 1 + \log N$  bits per snapshot per Nyquist-interval. Thus  $R_{NQ}(N) = O(\log N)$ .

2) *Distributed coding of raw data:* First, consider the *joint* compression of the raw binary data gathered by all sensors in a Nyquist-interval. By design, there is exactly one zero-crossing (whose direction is fixed) per Nyquist-interval per snapshot. Hence, the  $N$ -length vector of raw binary sensor data in any Nyquist-interval can take only  $N$  distinct values in each snapshot. Hence, even while ignoring the exact statistical structure of the temporal evolution, in the worst case, no more than  $\log_2(N)$  bits per snapshot per Nyquist-interval are needed to encode these  $N$ -length vectors (being the maximum entropy-rate of a source over an alphabet of size  $N$  [34]). This represents the worst-case *joint entropy-rate* (whenever the entropy rate exists) of all the (binary-valued) sensor outputs in any Nyquist-interval. However, there exist distributed source codes [33]–[35] which, by exploiting the underlying statistical correlation between sensor outputs, can achieve a compression efficiency of  $R_{NQ}(N) = \log_2(N)$  bits per Nyquist-interval per snapshot *without any sensor collaboration*. Further, if the joint statistics of all binary sensor data within each Nyquist-interval has a symmetric structure, then it is possible to noncollaboratively encode the data at each sensor at the *evenly distributed* per-sensor rate of  $R_{sensor}(N) = \frac{1}{N} \log_2(N)$  bits per snapshot.

## B. Information transport to CPU and Scaling Law

If the processed and coded data from each Nyquist-interval from each snapshot can be somehow moved to the CPU then the distortion (maximum expected point error) in reconstructing each temporal snapshot will go to zero as  $D = O(2^{-R_{NQ}}) = O(\frac{1}{N})$ . Equivalently, for a fixed  $D > 0$ , the number of one-bit sensors per Nyquist-interval needed will be finite with  $N \propto \frac{1}{D}$ . If the field is also temporally bandlimited and snapshots are taken at a stable temporal Nyquist-sampling rate then each temporal slice can be reconstructed to a precision  $O(\frac{1}{N})$ .<sup>10</sup> Hence, through temporal interpolation, the entire spatio-temporal field, that is, the field value at each point and at each time instant, can be reconstructed to a point-wise precision of  $O(\frac{1}{N})$ . Spatially bandlimited fields, however, are not spatially limited: there are an infinite number of Nyquist-intervals. A practical way to address this problem is to consider how the field reconstruction error in a compact

<sup>10</sup>Every lower-dimensional spatio-temporal slice of a bandlimited field is bandlimited to the same set of frequencies.

region of interest decays as the field is sampled over a growing neighborhood of sensor deployment.

1) *Field reconstruction in a compact region of interest:* Let  $[-\frac{L}{\lambda}, \frac{L}{\lambda}]$  be the interval of interest with the CPU located at the origin. A symmetric interval around the CPU is considered only for clarity and is not a restrictive assumption. The field is sampled in the larger interval  $[-\frac{L+L'}{\lambda}, \frac{L+L'}{\lambda}]$ . We want to analyze realizable rates of decay of reconstruction error with  $L'$ . Over each Nyquist-interval, consider  $N$  one-bit sensors spread uniformly as in Figure 5.

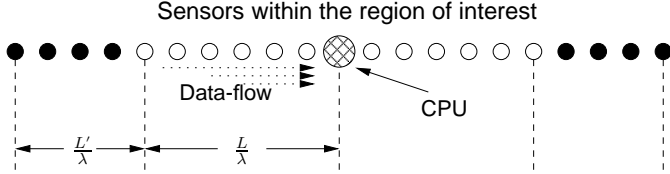


Fig. 5. The field is reconstructed over  $[-L/\lambda, L/\lambda]$  by sampling over a larger region  $[-(L+L')/\lambda, (L+L')/\lambda]$  to meet a desired reconstruction quality. Distortion  $D(N)$ , geographical size  $\frac{2}{\lambda}(L+L'(N))$ , and network rate  $R_{net}(N)$  are inter-related and depend on the number of one-bit sensors per Nyquist-length  $N$ . Links closer to the CPU carry greater (but always finite) traffic.

For each time snapshot of the field, let  $\mathcal{T} := \{t_l, l \in \mathcal{I}\}$  be the collection of locations corresponding to the mid-points of those inter-sensor intervals where the *first* sign-change is detected in each Nyquist-interval. Here,  $\mathcal{I} := \{-L-L', \dots, L+L'\}$ . As described in the previous subsection, the zero-crossings within each Nyquist-interval, can be located using neighbor-to-neighbor local communication or alternatively, the raw binary data at each sensor can be compressed using a distributed source code. Let  $P_l(t, \mathcal{T})$  denote the Lagrange interpolation polynomial for the  $l$ -th sampling location in  $\mathcal{T}$  [1]. Specifically,  $P_l(t, \mathcal{T}) = \prod_{j \neq l} (t - t_j)(t_l - t_j)^{-1}$ . Let

$$f_{\text{Lag}}(t) := \sum_{l \in \mathcal{I}} (-d(t_l)) P_l(t, \mathcal{T})$$

be the Lagrange-interpolated reconstruction of  $f(t)$  based on the samples at  $t_l \in \mathcal{T}$ , where  $d(t)$  is the dither field as defined in Section V. Thus  $f_{\text{Lag}}(t_l) = -d(t_l)$  for all  $l$ . In [11] it was shown that  $\forall t \in [-\frac{L}{\lambda}, \frac{L}{\lambda}]$ ,

$$|f(t) - f_{\text{Lag}}(t)| \leq \sqrt{L'} \left( \frac{\pi}{2\lambda} \right)^{2L'+1} + \frac{\tilde{C}}{N} L'^2, \quad (7.1)$$

if  $\lambda > \frac{\pi}{2}$ . The first error term in (7.1) arises from the finiteness of the sampling window and the second error term is due to quantization effects.

If  $L'(N)$  is chosen such that  $\left(\frac{\pi}{2\lambda}\right)^{2L'+1} = \frac{1}{N}$ , then  $L'(N) = \Theta(\log N)$  because  $\lambda > \frac{\pi}{2}$ . From (7.1) it follows that for this choice of  $L'(N)$ ,

$$D(N) := \sup_{t \in [-\frac{L}{\lambda}, \frac{L}{\lambda}]} |f(t) - f_{\text{Lag}}(t)| = O\left(\frac{(\log N)^2}{N}\right). \quad (7.2)$$

2) *Scaling Law for Multihopping:* Now suppose that the compressed data from each sensor (from a large number of snapshots) is routed through neighbor-to-neighbor local

communication in the direction of the CPU.<sup>11</sup> The node-id (address) information will be part of the header of each data packet sent to the CPU but will occupy only a vanishingly small fraction of the total data rate for a sufficiently large number of snapshots. The focus is on the sustainability of data rates with delays taken out of the picture. The total number of bits per snapshot generated by the *entire* network is given by

$$R_{net}(N) := 2(L + L'(N))R_{NQ}(N) = \Theta((\log N)^2), \quad (7.3)$$

where  $R_{NQ}(N) = (\log N)$  is the number of bits per Nyquist-interval per snapshot. Eliminating  $N$  from (7.2) and (7.3) gives

$$D(R_{net}) = O\left(R_{net} 2^{-\beta \sqrt{R_{net}}}\right), \quad (7.4)$$

where  $\beta = \sqrt{\log \frac{2\lambda}{\pi} - o(1)}$ . From (7.2), (7.3), and (7.4) it follows that any *nonzero* target distortion  $D > 0$  can be attained at the CPU provided that  $N$  (and  $L'(N)$ ) are sufficiently large (but finite) and the capacity of the busiest links (last-hop to the CPU) is of the order of  $R_{net}(N)$  bits per snapshot. The network traffic decreases approximately linearly with distance from the CPU. We summarize these results in terms of the following realizable scaling law.

**Multihop Scaling Law:** With  $N$  one-bit sensors per Nyquist-interval,  $\Theta(\log N)$  Nyquist-intervals, and total network bitrate  $R_{net} = \Theta((\log N)^2)$  (per-sensor bitrate  $\Theta((\log N)/N)$ ), the maximum pointwise distortion goes to zero as  $D = O((\log N)^2/N)$  or  $D = O(R_{net} 2^{-\beta \sqrt{R_{net}}})$ .

3) *Gaussian Multiple Access Uplink:* As an alternative to multihopping, suppose that a real Gaussian multiple access channel, bandlimited to  $W_C$  Hertz, can be realized for the effective multiuser uplink communication from all the sensors over  $\Theta(\log N)$  Nyquist-intervals to the CPU. Then the *sum capacity*, in bits per second, of the uplink channel grows as  $C_{sum}(W_C, P_{tot}) = W_C \log_2(1 + \Theta(P_{tot}(N)))$  [36], where  $P_{tot}$  is the sum power of all  $\Theta(N \log N)$  sensors. A sufficient condition, based on the separation of distributed source coding and multiuser channel coding, for successful transport of sensor data to the CPU is given by  $C_{sum} > R_{NQ} \times$  snapshots per second [34]. If  $W_C > 1$  and each sensor has a sufficiently large (but fixed) power, then  $P_{tot} = \Theta(N \log N)$ ,  $C_{sum} > R_{NQ}(N) \times$  snapshots per second, and successful data-transport to the CPU can be achieved with distortion decreasing as in (7.2).

4) *Discussion:* As  $N$  tends to infinity,  $D$  approaches zero. However, for a fixed, *nonzero* target distortion  $D$ , the number of sensors and the network rate needed is always *finite*. This should be contrasted with the following result discussed in [9] in an information-theoretic setting: if all sensors use *identical* deterministic scalar-quantizers then even if the sensor density approaches infinity, the field reconstruction MSE is bounded strictly away from zero. The results in [9] hold for all processes for which the probability of crossing some quantizer threshold is positive. This includes bounded WSS bandlimited processes studied in our work and also possibly

<sup>11</sup>A detailed description of the physical-layer and scheduling aspects of the local communication are beyond the scope of this work.



those which do not have a sufficient density of quantization threshold crossings for the reconstruction error to converge to zero as the sensor density increases to infinity [37]. The use of identical scalar quantizers can fundamentally limit the ability to decrease distortion by increasing sensor density. This is clearly seen by considering the extreme example of reconstructing a *constant* field using identical one-bit scalar quantizers. This suggests the need for having “diversity” in scalar quantization. Dithering is one way of achieving this diversity.

Also, as the desired distortion at the CPU approaches zero, one of two things will *necessarily* have to take place to sustain the desired performance: (i) If the CPU has only a finite number of links then the capacity of at least one of these links must approach infinity. This consequence is fundamental to rate-distortion theory and cannot be bypassed because for any continuous source and any reasonable distortion criterion, such as the space and time averaged mean squared error (MSE), zero distortion requires infinite rate. (ii) If the maximum capacity of each link is bounded then the number of links (independent communication channels) needed to sustain the desired level of reconstruction quality must go to infinity.

### C. Extension to two and higher spatial dimensions

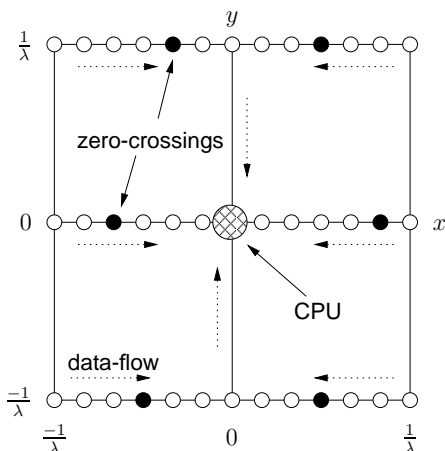


Fig. 6. **2-D distributed sampling of spatially bandlimited fields.** Horizontal dimension densely sampled:  $N$  one-bit sensors uniformly distributed over each horizontal Nyquist-interval. Vertical dimension sampled at the vertical Nyquist-period  $1/\lambda$ . Arrows indicate flow of compressed data towards the CPU. Distortion  $\downarrow 0$  as  $O(1/N)$  as the rate per Nyquist-area  $\uparrow$  as  $(\log N)$  like the 1-D case.

Although our focus has been on 1-D fields, the idea of Nyquist space-slices can be used to effect a simple extension of results (though by no means the best possible) from the 1-D one-bit dithered sampling framework (with multihopping) to two and higher dimensions. We only explain the essence of this idea, using Figure 6. The key observation is that it is sufficient to sample only one out of the many orthogonal spatial dimensions densely (and uniformly) using one-bit sensors while the other spatial dimensions are sampled at their respective Nyquist-rates. For example, for 2-D spatially bandlimited fields, one-bit sensors can be deployed along parallel

horizontal lines where the inter-line spacing is equal to the Nyquist-period along the vertical dimension. With  $N$  sensors spread uniformly over a horizontal Nyquist-length (rather than over an area), the number of sensors per Nyquist-area is also  $N$ . The field along each horizontal line corresponds to a horizontal spatial *slice* of the field and is therefore bandlimited to the same set of horizontal frequencies. Applying the results of the 1-D case it follows that the maximum reconstruction error of each Nyquist space-slice (and hence the entire spatial field) is of the order of  $O(1/N)$ . The bits per snapshot per Nyquist-area grows as  $(\log N)$ . If the field is also temporally bandlimited, snapshots can be taken at the temporal Nyquist-rate.

## VIII. CONCLUDING REMARKS

The problem of deterministic and stochastic distributed sampling of bandlimited sensor fields with low-precision ADCs was addressed in this work. The possibility of achieving any desired reconstruction quality using a multitude of low-precision sensors was shown using a dither-based sampling scheme. More importantly, the feasibility of flexible tradeoffs between the oversampling rate and the ADC-precision with respect to achieving exponential accuracy in the number of bits per Nyquist-interval was demonstrated. This exposed a key underlying conservation of bits principle. An achievable information scaling law for field reconstruction was also derived. Interestingly, this is possible using only neighbor-to-neighbor communication and distributed source coding, making it attractive for sensor networks. The extension of results from the deterministic to the stochastic setting in a strong almost-sure sense was also carried out. This paper is but a first step towards understanding the fundamentals of distributed sampling theory. Interesting research directions include extensions to fields with an unbounded dynamic range, e.g., Gaussian random fields, and addressing the effects of sensing noise, random deployment, location/timing errors and uncertainties, and synchronization and scheduling issues.

## ACKNOWLEDGMENT

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## APPENDIX I

### EXAMPLES OF AMPLITUDE LIMITED, BANDLIMITED STATIONARY PROCESSES

Two examples of stationary processes with uniformly amplitude limited sample paths and autocorrelation functions belonging to  $BL([-W, W])$  for some  $W > 0$  are presented. The first one is strictly stationary, the second is WSS.

**Example 1:** Let  $Z(t), t \in \mathbb{R}$ , be any strictly stationary process (e.g., any WSS Gaussian process). By passing  $Z$  through a hard-limiter (a pointwise operation),  $y(z) := \text{sign}(z) \min(|z|, 1)$ , one obtains a strictly stationary process  $Y(t), t \in \mathbb{R}$ , which is amplitude limited. Let  $X$  be the

process generated by passing  $Y$  through any BIBO-stable<sup>12</sup> linear, shift-invariant (LSI) system, with an impulse response belonging to  $BL([-W, W]) \cap \mathcal{L}^1(\mathbb{R})$ . Then  $X$  will (i) be amplitude limited (in every sample path), being the output of a BIBO stable LSI system with an amplitude limited input, (ii) stationary, being the output of an LSI system with a stationary input, and (iii) have an autocorrelation function belonging to  $BL([-W, W])$ , being the output of a bandlimited LSI system. **Example 2:** Let  $\{Y[l]\}_{l \in \mathbb{Z}}$  be any discrete, uniformly amplitude limited, WSS, white process, and  $\psi(t) := \left[ \frac{\sin(\pi t)}{\pi t} \right]^2$  for  $t \neq 0$  and  $\psi(0) := 1$ . Define a process

$$X(t) := \lim_{L \rightarrow \infty} \sum_{l=-L}^L Y[l] \psi(t+l+\Theta), \quad \forall t \in \mathbb{R},$$

where  $\Theta \sim \text{Uniform}[-1/2, 1/2]$  is independent of  $\{Y[l]\}_{l \in \mathbb{Z}}$ . Note that  $\psi$  is bandlimited, belongs to  $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ , and  $\sup_{t \in \mathbb{R}} \left\{ \sum_{l \in \mathbb{Z}} |\psi(t+l)| \right\} = C < \infty$ . Using this, it can be shown that  $X$  is a well defined WSS process, with a finite mean, and an autocorrelation function which belongs to  $BL([-W, W])$  and is uniformly amplitude limited. Moreover, every sample path of  $X$  is also uniformly amplitude limited.

## APPENDIX II

### PROOF OF PROPOSITION 3.1

$h(t)$  is differentiable at each  $t$ , and the derivative  $h' \in \mathcal{L}^1(\mathbb{R})$  because  $h'$  is uniformly amplitude limited and decays as  $\frac{1}{|t|^3}$ . Define  $f'(t) := \int_{\mathbb{R}} f(\tau) h'(t-\tau) d\tau$  and note that  $|f'(t)| \leq A \|h'\|_1 \leq A(W+W+\delta)(W+\delta) < \infty$ . Since  $f \in ZBL(W, \delta)$  for all  $\delta > 0$ , we have  $|f'(t)| \leq 2AW^2$ .

It remains to show that the derivative of  $f(t)$  is  $f'(t)$ . To see this, define  $h'_\epsilon(t) := \frac{1}{\epsilon} (h(t+\epsilon) - h(t))$ ,  $\epsilon \in (0, 1]$  and note that  $|h'_\epsilon(t) - h'(t)|$  goes to zero with decreasing  $\epsilon$  for each  $t \in \mathbb{R}$  and can be bounded from above by an absolutely integrable function  $g(t)$  that does not depend on  $\epsilon$ . Since  $|\frac{1}{\epsilon} (f(t+\epsilon) - f(t)) - f'(t)| \leq A \|h'_\epsilon - h'\|_1$ , the result follows from Lebesgue's Dominated Convergence Theorem [38, Theorem 1.34]. In fact, by using the same technique recursively, it can be demonstrated that  $f(t)$  is differentiable an arbitrary number of times and all derivatives are uniformly amplitude limited. ■

## APPENDIX III

### LIFTING SAMPLING AND INTERPOLATION RESULTS FROM $BL([-W, W])$ TO $ZBL(W, \delta)$

For all  $(t, \tau) \in \mathbb{R}^2$ , as  $L \uparrow \infty$  along positive integers,

$$\begin{aligned} \nu_L(t, \tau) &:= \sum_{l=-L}^L |f(\tau) h(t_l - \tau) \psi_l(t - t_l)| \quad \uparrow \\ \nu(t, \tau) &:= \sum_{l=-\infty}^{\infty} |f(\tau) h(t_l - \tau) \psi_l(t - t_l)|. \end{aligned}$$

Since  $\|f\|_\infty, \|h\|_\infty < \infty$  and for all  $(t, L) \in \mathbb{R} \times \mathbb{N}$ ,  $\sum_{l=-L}^L |\psi_l(t - t_l)| \leq C < \infty$ ,

$$0 \leq \nu(t, \tau) \leq \|f\|_\infty \|h\|_\infty C < \infty.$$

Also, since  $\|h\|_1 < \infty$ , for all  $(t, L) \in \mathbb{R} \times \mathbb{N}$ ,

$$\int_{\mathbb{R}} \nu_L(t, \tau) d\tau \leq C \|f\|_\infty \|h\|_1 < \infty.$$

To summarize the above in words,  $\nu(t, \tau)$  is a nonnegative, uniformly amplitude limited, measurable field and for all  $(t, \tau) \in \mathbb{R}^2$ ,  $\nu_L(t, \tau)$  is a nondecreasing sequence of nonnegative real numbers converging to  $\nu(t, \tau)$ . For each  $(t, L) \in \mathbb{R} \times \mathbb{N}$ ,  $\nu_L(t, \tau)$  is a measurable, absolutely-integrable function of  $\tau$ . Hence, by Lebesgue's Monotone Convergence Theorem [38, Theorem 1.26], for all  $t \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} \nu(t, \tau) d\tau = \lim_{L \rightarrow \infty} \int_{\mathbb{R}} \nu_L(t, \tau) d\tau \leq C \|f\|_\infty \|h\|_1.$$

Thus, for all  $t \in \mathbb{R}$ ,  $\nu(t, \tau)$  is a measurable, absolutely-integrable function of  $\tau$ . For all  $(t, \tau, L) \in \mathbb{R}^2 \times \mathbb{N}$  let

$$g_L(t, \tau) := f(\tau) \sum_{l=-L}^L h(t_l - \tau) \psi_l(t - t_l).$$

By assumption (ii) in the Lemma and because  $\|f\|_\infty < \infty$ , as  $L \uparrow \infty$  along positive integers, for all  $(t, \tau) \in \mathbb{R}^2$ ,

$$g_L(t, \tau) \rightarrow f(\tau) h(t - \tau).$$

Furthermore, for all  $(t, \tau) \in \mathbb{R}^2$ ,

$$|g_L(t, \tau)| \leq \nu(t, \tau),$$

that is, for each  $t \in \mathbb{R}$ ,  $g_L(t, \tau)$  is dominated by  $\nu(t, \tau)$  which is an absolutely-integrable function of  $\tau$ . Hence, by Lebesgue's Dominated Convergence Theorem [38, Theorem 1.34],

$$\int_{\mathbb{R}} f(\tau) h(t - \tau) d\tau = \lim_{L \rightarrow \infty} \int_{\mathbb{R}} g_L(t, \tau) d\tau, \quad (\text{III.1})$$

for all  $t \in \mathbb{R}$ . Since  $f(\tau)$  is a Zakai-sense bandlimited function of  $\tau$ , with associated Toeplitz reproducing kernel  $h(\tau)$ , from (3.1) we have for all  $t \in \mathbb{R}$ ,

$$f(t) = \int_{\mathbb{R}} f(\tau) h(t - \tau) d\tau \quad (\text{III.2})$$

and also

$$\begin{aligned} \int_{\mathbb{R}} g_L(t, \tau) d\tau &= \sum_{l=-L}^L \psi_l(t - t_l) \int_{\mathbb{R}} f(\tau) h(t_l - \tau) d\tau \\ &= \sum_{l=-L}^L f(t_l) \psi_l(t - t_l). \end{aligned} \quad (\text{III.3})$$

Hence, from (III.1), (III.2), and (III.3), we have shown that for all  $t \in \mathbb{R}$ ,

$$f(t) = \lim_{L \rightarrow \infty} \sum_{l=-L}^L f(t_l) \psi_l(t - t_l). \quad (\text{III.4})$$

Since  $\|f\|_\infty < \infty$  and for all  $(t, L) \in \mathbb{R} \times \mathbb{N}$ ,  $\sum_{l=-L}^L |\psi_l(t - t_l)| \leq C < \infty$ , the series in (III.4) converges absolutely and uniformly on all compact subsets of  $\mathbb{R}$ . This completes the proof of the extension of sampling and interpolation results from  $BL([-W, W])$  to  $ZBL(W, \delta)$ . ■

<sup>12</sup>BIBO stands for bounded-input-bounded-output.

APPENDIX IV  
DESIGN OF  $b$ -BIT DITHER FIELDS

There are several possible approaches for designing a  $b$ -bit dither field. Here we present only a simple natural construction based on dither fields for one-bit sampling ( $b = 1$ ). For any  $1 < b < k$ , let  $M := 2^{b-1}$ . For the design, it is sufficient to consider the single interval  $[A_0, B_0] = [0, \frac{1}{M\lambda} - \tau]$ ,  $\tau = 1/(2^k\lambda)$ , i.e., the interval next to the origin corresponding to  $l = 0$  in Section VI. We define a  $b$ -bit dither field as follows

$$d_b(t) = \frac{2c}{M\gamma} d(t/(\lambda B_0)), \quad (IV.1)$$

where  $c > (1+\pi)/4$  is any constant and  $d(t)$  is any dither field used in the one-bit dithered sampling scheme of Section V with maximum amplitude  $\gamma > 1$  and a derivative whose amplitude is bounded by  $\Delta$ . From Section V-A we recall that  $d(0) = -d(1/\lambda) = \pm\gamma$ . We may assume, without loss of generality, that  $d(0) > 0$  so that  $d_b(A_0) = -d_b(B_0) = 2c/M > 2/M$ . Also note that  $|d'_b(t)| \leq \frac{2c}{M\gamma\lambda B_0} \Delta = \frac{2c\Delta}{\gamma} (1 - \frac{2^{b-1}}{2^k})^{-1} < \frac{4c}{\gamma} \Delta$  because  $1 < b < k$ . Thus,  $\bar{\Delta} := \sup |d'_b(t)| < \frac{4c}{\gamma} \Delta$ .

It will be shown that the dithered-field  $[f + d_b](t)$  crosses a level in the set  $\{0, \pm\frac{1}{M}, \dots, \pm\frac{M-1}{M}\}$  in the interval  $[A_0, B_0]$ . By design,  $d_b(t)$  is continuous. The intermediate value theorem for continuous functions is used to prove the existence of a level-crossing. Specifically, it will be shown that  $[f + d_b](A_0)$  and  $[f + d_b](B_0)$  lie on different sides of some level. Let  $L_j, -M \leq j \leq (M-1)$ , be quantization intervals given by

$$L_j = \begin{cases} (-\infty, -\frac{M-1}{M}) & \text{for } j = -M, \\ [\frac{j}{M}, \frac{j+1}{M}) & \text{for } -(M-1) \leq j \leq (M-2), \\ [\frac{M-1}{M}, \infty) & \text{for } j = (M-1). \end{cases}$$

Since  $d_b(A_0) > \frac{2}{M}$ , therefore,  $f(A_0) + d_b(A_0) > -1 + \frac{2}{M}$ . Thus,  $[f + d_b](A_0) \notin L_j$  for  $j \leq -(M-1)$ . So let  $[f + d_b](A_0) \in L_j$  for some  $j$ ,  $-(M-2) \leq j \leq (M-2)$ . We now show that  $[f + d_b](B_0)$  lies in a different quantization interval than  $L_j$ . We have the following inequalities

$$\begin{aligned} [f + d_b](B_0) &= f(B_0) - \frac{2c}{M}, \\ &\stackrel{(i)}{\leq} f(A_0) + \pi(B_0 - A_0) - \frac{2c}{M}, \\ &\stackrel{(ii)}{<} \frac{j+1}{M} - \frac{2c}{M} + \pi(B_0 - A_0) - \frac{2c}{M}, \\ &\stackrel{(iii)}{\leq} \frac{j}{M}, \end{aligned}$$

where (i) follows from the mean value theorem applied to  $f(t)$  at the end points  $A_0$  and  $B_0$  and noting that  $|f'(t)|$  is bounded by  $\pi$  (Bernstein's inequality), (ii) is because  $[f + d_b](A_0) = f(A_0) + \frac{2c}{M} \in L_j, -(M-2) \leq j \leq (M-2), \Rightarrow f(A_0) + \frac{2c}{M} < \frac{j+1}{M}$ , and (iii) follows from  $c > (1+\pi)/4$ ,  $\lambda > 1$ , and some algebra. Since  $[f + d_b](B_0) < j/M$ ,  $[f + d_b](B_0) \in L_{j'}$  for some  $j' \leq (j-1)$ , i.e.,  $[f + d_b](B_0) \notin L_j$ . This together with the assumption that  $[f + d_b](A_0) \in L_j, -(M-2) \leq j \leq (M-2)$  shows that there is a level-crossing of  $[f + d_b](t)$  in the interval  $[A_0, B_0]$ .

In the special case when  $[f + d_b](A_0) \in L_{(M-1)}$ , then  $[f + d_b](B_0) = f(B_0) - 2c/M < 1 - 2/M$ . Hence,  $[f + d_b](B_0)$

lies in the quantization interval  $L_i, i \leq (M-3)$  and the level  $(M-1)/M$  will be crossed by the dithered field.

REFERENCES

- [1] R. J. Marks, II, *Introduction to Shannon Sampling and Interpolation Theory*. New York, NY: Springer-Verlag, 1990.
- [2] J. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*. USA: Clarendon Press, 1996.
- [3] N. T. Thao and M. Vetterli, "Reduction of the MSE in R-times oversampled A/D conversion  $O(1/R)$  to  $O(1/R^2)$ ," *IEEE Trans. Signal Proc.*, vol. 42, no. 1, pp. 200–203, Jan. 1994.
- [4] —, "Deterministic Analysis of Oversampled A/D Conversion and Decoding Improvement Based on Consistent Estimates," *IEEE Trans. Signal Proc.*, vol. 42, no. 3, pp. 519–531, Mar. 1994.
- [5] Z. Cvetković and M. Vetterli, "Error-rate Characteristics of Oversampled Analog-to-Digital Conversion," *IEEE Transactions on Information Theory*, vol. 44, no. 5, pp. 1961–1964, Sep. 1998.
- [6] F. A. Marvasti, Ed., *Nonuniform Sampling: Theory and Practice*. Kluwer Academic/Plenum Publishers, Jun. 2001.
- [7] A. Kashyap, L. A. Lastras-Montano, C. Xia, and Z. Liu, "Distributed source coding in dense sensor networks," in *Proceedings of the Data Compression Conference (DCC)*, Snowbird, UT, Apr. 2005.
- [8] D. L. Neuhoff and S. S. Pradhan, "An upper bound to the rate of ideal distributed lossy source coding of densely sampled data," in *Proc. IEEE International Conference on Acoustics, Speech and Signal Processing*, Toulouse, France, May 2006, pp. 1137–1140.
- [9] D. Marco, E. J. Duarte-Melo, M. Liu, and D. L. Neuhoff, "On the Many-to-One Transport Capacity of a Dense Wireless Sensor Network and the Compressibility of its Data," in *Proc. 2nd Intl. Wkshp. Information Processing in Sensor Networks*, Palo Alto, CA, Apr. 2003, pp. 1–16.
- [10] P. Gupta and P. R. Kumar, "The Capacity of Wireless Networks," *IEEE Transactions on Information Theory*, vol. IT-46, no. 2, pp. 388–404, Mar. 2000.
- [11] Z. Cvetković and I. Daubechies, "Single Bit oversampled A/D conversion with exponential accuracy in bit rate," *Proc. DCC*, pp. 343–352, Mar. 2000.
- [12] M. Zakai, "Band-Limited Functions and the Sampling Theorem," *Information and Control*, 8 (1965), pp. 143–158.
- [13] S. Cambanis and E. Masry, "Zakai's class of Bandlimited Functions and Processes: Its Characterization and Properties," *SIAM Journal on Applied Mathematics*, Vol. 30, No. 1, Jan. 1976, pp. 10–21.
- [14] A. Kumar, P. Ishwar, and K. Ramchandran, "On distributed sampling of smooth non-bandlimited fields," in *Proc. Third Intl. Symposium Information Processing in Sensor Networks*. New York, NY: ACM Press, 2004, pp. 89–98.
- [15] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*. London, UK: Cambridge University Press, 1959.
- [16] A. Papoulis, *Signal Analysis*. McGraw-Hill, New York, NY, 1977.
- [17] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. USA: McGraw-Hill Companies, 1965.
- [18] Yu. K. Belyaev, "Analytic Random Processes," *Theory of Probability and its Applications*, Vol. IV, No. 4, 1959, eds. N. Artin and K. L. Chung, pub. SIAM, 1961, pp. 402–409. (English translation of Soviet journal: Teoriya Veroyatnostei i ee Primeneniya by R. A. Silverman).
- [19] R. M. Gray and D. L. Neuhoff, "Quantization," *IEEE Transactions on Information Theory*, vol. IT-44, pp. 2325–2383, Oct. 1998.
- [20] I. Daubechies, R. A. DeVore, S. Güntürk, and V. Vaishampayan, "A/D Conversion with Imperfect Quantizers," *IEEE Transactions on Information Theory*, vol. 52, no. 3, pp. 874–885, Mar. 2006.
- [21] R. Durrett, *Probability: Theory and Examples*, 2nd ed. Belmont, CA: Duxbury Press, 1996.
- [22] R. M. Gray, "Oversampled Sigma-Delta Modulation," *IEEE Trans. Commun.*, vol. 35, no. 5, pp. 481–489, May 1987.
- [23] E. Masry and S. Cambanis, "Consistent Estimation of Continuous-Time Signals from Nonlinear Transformations of Noisy Samples," *IEEE Transactions on Information Theory*, vol. 27, no. 1, pp. 84–96, Jan. 1981.
- [24] E. Masry, "The Reconstruction of Analog Signals from the Sign of Their Noisy Samples," *IEEE Transactions on Information Theory*, vol. 27, no. 6, pp. 735–745, Nov. 1981.
- [25] F. A. Marvasti, *Unified Approach to Zero-Crossing and Nonuniform Sampling for Single and Multidimensional Signals and Systems*. Non Uniform, May 1987.
- [26] A. I. Zayed, *Advances in Shannon's Sampling Theory*. CRC Press, Boca Raton, FL, 1993.



- [27] R. Benzi, A. Sutera, and A. Vulpiani, "The Mechanism of Stochastic Resonance," *J. Phys. A*, vol. 14, pp. L453-L457, 1981.
- [28] A. R. Bulsara and L. Gammaioni, "Tuning in to Noise," *Phys. Today*, vol. 49, pp. 39-45, Mar. 1996.
- [29] F. Moss, L. M. Ward, and W. G. Sannita, "Stochastic resonance and sensory information processing: a tutorial and review of applications," *Clinical and Neurophysiology*, Feb. 2004, pp. 267-281.
- [30] V. Pascazio and G. Schirinzi, "Synthetic aperture radar imaging by one bit coded signals," *Electronics and Communication Engineering Journal*, vol. 10, no. 1, pp. 17-28, Feb. 1998.
- [31] P. Franceschetti, M. Tesauro, and P. A. Rosen, "One-bit coding for Spot-Light SAR: applications, algorithms and architecture implementation," in *Proc. IEEE International Geoscience and Remote Sensing Symposium (IGARSS)*, vol. 4, Jun. 1999, pp. 2273-2275.
- [32] W. Rudin, *Principles of Mathematical Analysis*. USA: McGraw-Hill Companies, 1976.
- [33] D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Transactions on Information Theory*, vol. 19, pp. 471-480, Jul. 1973.
- [34] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York, NY: John Wiley and Sons, 1991.
- [35] S. S. Pradhan and K. Ramchandran, "Distributed source coding using syndromes (DISCUS): Design and construction," *IEEE Transactions on Information Theory*, vol. 49, no. 3, pp. 626-643, Mar. 2003.
- [36] S. V. Hanly and D. N. Tse, "Multi-Access Fading Channels: Shannon and Delay-Limited Capacities," in *33rd annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, Oct. 1995.
- [37] Z. Cvetković and M. Vetterli, "On Simple Oversampled A/D Conversion in  $L^2(\mathbb{R})$ ," *IEEE Transactions on Information Theory*, vol. 47, no. 1, pp. 59-73, Jan. 2001.
- [38] W. Rudin, *Real and Complex Analysis*. 3rd ed., McGraw-Hill, 1987.